

## EQUIVARIANT FIXED POINT THEORY

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## ABSTRACT

The Lefschetz fixed point theorem and its converse have many generalizations. One of these generalizations is to endomorphisms of a space with a group action. In this paper we define equivariant Lefschetz numbers and Reidemeister traces using traces in bicategories with shadows. We use the functoriality of this trace to identify different forms of these invariants and to prove an equivariant Lefschetz fixed point theorem and its converse.

## INTRODUCTION

The Lefschetz fixed point theorem gives an easily calculable sufficient condition for a continuous endomorphism of a closed smooth manifold to have a fixed point. This theorem has many generalizations. One of these generalization is the equivariant Lefschetz fixed point theorem.

**Theorem A** (Equivariant Lefschetz fixed point theorem). *Let  $G$  be a finite group,  $X$  be a closed smooth  $G$ -manifold, and*

$$f: X \rightarrow X$$

*be an equivariant endomorphism. If  $f$  is equivariantly homotopic to a map with no fixed points then the equivariant Lefschetz number of  $f$  is zero.*

The equivariant Lefschetz number is a generalization of the classical Lefschetz number. It is defined in terms of the classical Lefschetz numbers of the subspaces of  $X$  fixed by subgroups of  $G$ . In this form, this theorem was proved by Lück and Rosenberg in [21].

The classical and equivariant Lefschetz fixed point theorems do not give necessary conditions for an endomorphism to have a fixed point. There is a refinement of the Lefschetz number that gives a necessary and sufficient condition if certain dimension conditions are satisfied. The refinement of the Lefschetz number is called the Reidemeister trace.

**Theorem B** (Converse to the equivariant Lefschetz fixed point theorem). *Let  $X$  be a closed smooth  $G$ -manifold such that*

$$\dim(X^H) \geq 3 \text{ and } \dim(X^H) \leq \dim(X^K) - 2$$

*for all subgroups  $K \subset H$  of  $G$  that are isotropy groups of  $X$ . Then an equivariant endomorphism*

$$f: X \rightarrow X$$

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*is equivariantly homotopic to a map with no fixed points if and only if the equivariant Reidemeister trace of  $f$  is zero.*

Version of this theorem have been proved by Fadell and Wong [9], Wong [35], Weber [32], and Klein and Williams [15].

The original proofs of Theorems A and B were generalizations of the corresponding proofs of the classical case where  $G$  is the trivial group. These proofs used simplicial techniques and were difficult to generalize. In [8] an alternative approach to the proof of the classical version of Theorem A was developed. This approach is more conceptual and is independent of the previous proofs. In [16] and [24] a corresponding approach to the classical version of Theorem B was developed. In this paper we will describe how to apply this approach to the proofs of Theorems A and B.

A significant part of both of these theorems is the identification of different descriptions of the same invariant. In [24] we defined the trace in a bicategory with shadows and showed that it can be used to prove these identifications in the classical cases. Here we show our approach is equally applicable to equivariant invariants. This gives independent proofs of Theorems A and B that are immediate generalizations of the proofs of the corresponding classical results.

The approach of trace in bicategories captures many of the known equivariant fixed point invariants. These include the equivariant fixed point indices in [8, 21], the equivariant Lefschetz number in [17, 33], the equivariant Nielsen number in [9, 34], and the equivariant Reidemeister traces in [31, 32]. This approach also describes new invariants. Some of these new invariants detect fixed orbits in addition to fixed points.

In the first two sections of this paper we recall the definitions of traces in symmetric monoidal categories and bicategories with shadows. In Sections 3 and 4 we describe two examples of bicategories with shadows. In Sections 5 through 8 we prove Theorem A. In Sections 9 through 14 we prove Theorem B.

We assume the reader is familiar with the classical fixed point index described in [3, 7] and Nielsen theory described in [3, 13].

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**Notation.**  $G$  denotes a finite group and  $H$  is a subgroup of  $G$ .  $NH$  is the normalizer of  $H$  in  $G$ ,  $WH$  is  $NH/H$ .  $(H)$  denotes the conjugacy class of the subgroup  $H$  in  $G$ .

If  $X$  is a right  $G$ -space and  $x \in X$ ,  $G_x := \{g \in G | xg = x\}$ ,

$$X^H := \{x \in X | xh = x \text{ for all } h \in H\}$$

$$X_H := \{x \in X | G_x = H\}$$

$$X^{>H} := \{x \in X | H \subsetneq G_x\}$$

$$X_{(H)} := \{x \in X | (G_x) = (H)\}$$

$$X^{(H)} := \{x \in X | (H) \subset (G_x)\}$$

If  $x: G/H \rightarrow X$  is a  $G$ -map,  $X_H(x)$  is the component of  $X_H$  that contains  $x(eH)$  and  $X^H(x)$  is the component of  $X^H$  that contains  $x(eH)$ .

**Topological assumptions.** We will focus on  $G$ -manifolds and  $G$ -ENR's. This assumption allows us to consider only particularly nice maps without any loss of generality.

For the definition of a  $G$ -ENR see [12].

**Definition 0.1.** [10, 4.3] Let  $X$  and  $Y$  be  $G$ -spaces. A  $G$ -map  $f: X \rightarrow Y$  is *taut* if for all isotropy subgroups  $H$  of  $X$  there is a neighborhood  $V$  of  $X^{>H}$  in  $X^H$  and an equivariant retraction  $r_H: V \rightarrow X^{>H}$  such that  $f^H|_V = f^H \circ r_H$ .

**Proposition 0.2.** [10, 4.4][33, 2.5] *Let  $X$  be a compact  $G$ -ENR and  $f: X \rightarrow X$  a continuous  $G$ -map. Then  $f$  is  $G$ -homotopic to a taut map.*

This proposition is a consequence of the following observation.

**Proposition 0.3.** [30, II.1.9, II.6.7] *If  $X$  is a  $G$ -ENR the inclusion*

$$X^{>H} \rightarrow X^H$$

*is a  $WH$ -cofibration for any subgroup  $H$  of  $G$ .*

We will assume maps are taut. If a map is not taut, we will replace it by an equivariantly homotopic map that is taut. Since all the invariants defined here are invariants of  $G$ -homotopy classes the choice does not matter.

Let  $F_g$  be the fixed points<sup>1</sup> of the map  $f \cdot g$ .

**Proposition 0.4.** *Let  $f$  be a taut map and  $g$  and  $h$  be elements of  $NH$  that are not conjugate in  $WH$ . Then there are open neighborhoods  $U_g$  of  $F_g \cap X_{(H)}$  and  $U_h$  of  $F_h \cap X_{(H)}$  in  $X^{(H)}$ , disjoint from  $X^{>(H)}$ , such that  $U_g \cap U_h$  and  $(U_g/WH) \cap (U_h/WH)$  are empty.*

Since  $G$  is finite, for each isotropy class of subgroups  $H$  of  $G$  we use this proposition to choose pairwise disjoint neighborhoods of the fixed points of  $f \cdot g$  for a representative  $g$  of each conjugacy class in  $WH$ .

*Proof.* First note that for any taut map  $f: U \rightarrow X$  and every conjugacy class of subgroups  $(H)$  of  $G$

$$\overline{F_e \cap U_{(H)}} \cap U^{>(H)}$$

is empty. See [22, 5.3] for a proof. Since  $f \cdot g$  is also taut,

$$\overline{F_g \cap U_{(H)}} \cap U^{>(H)}$$

is empty for each  $g$ .

The action of  $WH$  on  $X_H$  is free, so  $F_g \cap F_h \cap X_H$  and  $(F_g/WH) \cap (F_h/WH) \cap (X_H/WH)$  are empty if  $g$  and  $h$  are not conjugate in  $WH$ . This implies that

$$\overline{F_g \cap X_H} \cap (F_h \cap X_H)$$

and

$$\overline{(F_g/WH) \cap (X_H/WH)} \cap ((F_h/WH) \cap (X_H/WH))$$

are also empty.

The sets  $F_g \cap X_H$ ,  $F_h \cap X_H$  and  $X^{>H}$  are pairwise separated. Since  $X$  is completely normal there are pairwise disjoint open neighborhoods of these sets. Similarly, the sets  $(F_g/WH) \cap (X_H/WH)$ ,  $(F_h/WH) \cap (X_H/WH)$ , and  $X^{>H}/WH$

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<sup>1</sup>In this paper “fixed point” means fixed with respect to a map, i.e.  $f(x) = x$ . It does **not** necessarily mean fixed with respect to the group action.

are pairwise separated. The disjoint neighborhoods of the lemma are the neighborhoods chosen above modified by the neighborhoods of  $(F_g/WH) \cap (X_H/WH)$  and  $(F_h/WH) \cap (X_H/WH)$ .  $\square$

## 1. DUALITY AND TRACE IN SYMMETRIC MONOIDAL CATEGORIES

The trace in symmetric monoidal categories is a generalization of the trace in linear algebra that retains many important properties. In particular, it satisfies a generalization of invariance of basis and is functorial. The generalized trace defines a trace for endomorphisms of modules over a commutative ring, endomorphisms of chain complexes of modules over a commutative ring, and endomorphisms of closed smooth manifolds.

There are several sources for additional details on this trace. These include [8, 19, 26].

Let  $\mathcal{V}$  be a symmetric monoidal category with monoidal product  $\otimes$ , unit  $S$ , and symmetry isomorphism

$$\gamma: A \otimes B \rightarrow B \otimes A.$$

**Definition 1.1.** An object  $A$  in  $\mathcal{V}$  is *dualizable* with dual  $B$  if there are morphisms

$$\eta: S \rightarrow A \otimes B$$

and

$$\epsilon: B \otimes A \rightarrow S$$

such that the composites

$$A \cong S \otimes A \xrightarrow{\eta \otimes \text{id}} A \otimes B \otimes A \xrightarrow{\text{id} \otimes \epsilon} A \otimes S \cong A$$

and

$$B \cong B \otimes S \xrightarrow{\text{id} \otimes \eta} B \otimes A \otimes B \xrightarrow{\epsilon \otimes \text{id}} S \otimes B \cong B$$

are the identity maps of  $A$  and  $B$  respectively.

The map  $\eta$  is called the *coevaluation* and  $\epsilon$  is called the *evaluation*.

The category of modules over a commutative ring  $R$  is a symmetric monoidal category. The tensor product over  $R$  is the monoidal product. A module  $M$  is dualizable if and only if it is a finitely generated projective  $R$ -module. In this case, the dual of  $M$  is  $\text{Hom}_R(M, R)$ . The evaluation map,

$$\epsilon: \text{Hom}_R(M, R) \otimes_R M \rightarrow R,$$

is  $\epsilon(\phi, m) = \phi(m)$ . Since  $M$  is finitely generated and projective, the dual basis theorem implies there is a ‘basis’  $\{m_1, m_2, \dots, m_n\}$  with dual ‘basis’  $\{m'_1, m'_2, \dots, m'_n\}$ . The coevaluation is defined by linearly extending

$$\eta(1) = \sum_{i=1}^n m_i \otimes m'_i.$$

**Definition 1.2.** If  $A$  is dualizable with dual  $B$  and  $f: A \rightarrow A$  is an endomorphism in  $\mathcal{V}$ , the *trace* of  $f$ ,  $\text{tr}(f)$ , is the composite

$$S \xrightarrow{\eta} A \otimes B \xrightarrow{f \otimes \text{id}} A \otimes B \xrightarrow{\gamma} B \otimes A \xrightarrow{\epsilon} S.$$

If  $R$  is a field, a finitely generated projective  $R$ -module  $M$  is a finite dimensional vector space. The trace of an endomorphism is the sum of the diagonal elements in a matrix representation.

Let  $A$  be a dualizable object of  $\mathcal{V}$ ,  $C$  be an object of  $\mathcal{V}$ , and  $\Delta: A \rightarrow C \otimes A$  be a morphism in  $\mathcal{V}$ .

**Definition 1.3.** If  $f: A \rightarrow A$  is a morphism in  $\mathcal{V}$  the *transfer* of  $f$  with respect to  $\Delta$ ,  $\text{tr}_\Delta(f)$ , is the composite

$$\begin{array}{ccccc} S & \xrightarrow{\eta} & A \otimes B & \xrightarrow{\Delta \otimes \text{id}} & C \otimes A \otimes B \\ & & & \downarrow \text{id} \otimes f \otimes \text{id} & \\ & & C \otimes A \otimes B & \xrightarrow{\text{id} \otimes \gamma} & C \otimes B \otimes A \xrightarrow{\text{id} \otimes \epsilon} C \otimes S \cong C. \end{array}$$

The equivariant stable homotopy category is a symmetric monoidal category. We say a space is dualizable if its suspension spectrum is dualizable. We can also give another description of these duals.

Let  $V$  be a  $G$ -representation and  $S^V$  be the 1-point compactification of  $V$ . The base point of  $S^V$  is the point at infinity.

**Definition 1.4.** [19, III.3.5] A based  $G$ -space  $X$  is  $V$ -dualizable if there is a based  $G$ -space  $Y$  and equivariant maps  $\eta: S^V \rightarrow X \wedge Y$  and  $\epsilon: Y \wedge X \rightarrow S^V$  such that the diagrams

$$\begin{array}{ccc} S^V \wedge X & \xrightarrow{\eta \wedge \text{id}} & X \wedge Y \wedge X \\ & \searrow \gamma & \downarrow \text{id} \wedge \epsilon \\ & & X \wedge S^V \end{array} \quad \begin{array}{ccc} Y \wedge S^V & \xrightarrow{\text{id} \wedge \eta} & Y \wedge X \wedge Y \\ & \searrow (\sigma \wedge \text{id})\gamma & \downarrow \epsilon \wedge \text{id} \\ & & S^V \wedge Y \end{array}$$

commute up to equivariant stable homotopy.

The map  $\sigma$  is defined by  $\sigma(v) = -v$ .

An orbit  $G/H_+$  is dualizable for any subgroup  $H$  of  $G$  and the dual is  $G/H_+$ . There are two other examples where we can give explicit descriptions of the dual.

**Proposition 1.5.** [19, III.4.1, III.5.1]

- (1) If  $X$  is a compact  $G$ -ENR that embeds in a representation  $V$ ,  $X_+$  is  $V$ -dualizable with dual the cone on the inclusion  $V \setminus X \rightarrow V$ .
- (2) If  $M$  is a closed smooth  $G$ -manifold that embeds in a representation  $V$ , then  $M_+$  is  $V$ -dualizable with dual  $T\nu$ , the Thom space of the normal bundle of the embedding of  $M$  in  $V$ .

The trace of an endomorphism of a  $G$ -space regarded as a map in the equivariant stable homotopy category is called the *local Lefschetz number*. The local Lefschetz number is the stable homotopy class of a map

$$S^V \rightarrow S^V$$

for some representation  $V$  and so is an element of the  $0^{\text{th}}$  equivariant stable homotopy group of  $S^0$ ,  $\pi_0^{s,G}$ . Using this definition of the local Lefschetz number, it is clear that the local Lefschetz number is an invariant of the equivariant homotopy class. This definition can be generalized to maps

$$f: U \rightarrow X$$

where  $U$  is an open invariant subspace of  $X$  and the set of fixed points of  $f$  is compact. See [30, III.1.4]. The local Lefschetz number is also called the fixed point index.

The local Lefschetz number of the identity map of a  $G$ -space  $X$  is called the *equivariant Euler characteristic* of  $X$ . It is denoted  $\chi(X)$ . This definition is consistent with the classical definition of the Euler characteristic.

**Proposition 1.6** (Additivity of trace and transfer). [30, III.5.3][27, 1.17] *Let  $U$  be an open invariant subset of  $X$ ,  $f: U \rightarrow X$  be an equivariant map, and  $V$  and  $V'$  be open invariant subsets of  $X$  such that  $V \cup V' = U$  and  $V \cap V'$  contains no fixed points.*

*Then*

$$\mathrm{tr}(f|_V) + \mathrm{tr}(f|_{V'}) = \mathrm{tr}(f).$$

*If  $\iota_V$  and  $\iota_{V'}$  are the inclusions of  $V$  and  $V'$  into  $X$  and  $\Delta$  is the diagonal map*

$$(\iota_V)_*(\mathrm{tr}_\Delta(f|_V)) + (\iota_{V'})_*(\mathrm{tr}_\Delta(f|_{V'})) = \mathrm{tr}_\Delta(f).$$

## 2. DUALITY AND TRACE IN BICATEGORIES WITH SHADOWS

The trace in a symmetric monoidal category describes the classical and equivariant local Lefschetz number and the classical (global) Lefschetz number. It cannot be used to describe the refinements of the Lefschetz number to the Reidemeister trace. It also cannot be used to define the equivariant Lefschetz numbers defined in [17, 21]. To describe these invariants we need to use the trace in a bicategory with shadows.

This section is a brief summary of the relevant parts of [23, 24, 26]. Here we define duality and trace in bicategories with shadows and state some basic properties. We will give applications of this trace in Sections 6, 7, 10, 11, and 12.

**Definition 2.1.** [18, 1.0] A *bicategory*  $\mathcal{B}$  consists of

- (1) A collection  $\mathrm{ob}\mathcal{B}$ .
- (2) Categories  $\mathcal{B}(A, B)$  for each  $A, B \in \mathrm{ob}\mathcal{B}$ .
- (3) Functors

$$\odot: \mathcal{B}(A, B) \times \mathcal{B}(B, C) \rightarrow \mathcal{B}(A, C)$$

$$U_A: * \rightarrow \mathcal{B}(A, A)$$

for  $A, B$  and  $C$  in  $\mathrm{ob}\mathcal{B}$ .

Here  $*$  denotes the category with one object and one morphism. The functors  $\odot$  are required to satisfy unit and associativity axioms up to natural isomorphism 2-cells.

The elements of  $\mathrm{ob}\mathcal{B}$  are called *0-cells*. The objects of  $\mathcal{B}(A, B)$  are called *1-cells*. The morphisms of  $\mathcal{B}(A, B)$  are called *2-cells*.

There is a bicategory, denoted  $\mathbf{Mod}$ , with 0-cells rings, 1-cells bimodules, and 2-cells homomorphisms. The bicategory composition is tensor product.

**Definition 2.2.** [23, 16.4.1] A 1-cell  $X \in \mathcal{B}(A, B)$  is *right dualizable* with dual  $Y \in \mathcal{B}(B, A)$  if there are 2-cells

$$\eta: U_A \longrightarrow X \odot Y \quad \epsilon: Y \odot X \longrightarrow U_B$$

such that

$$Y \cong Y \odot U_A \xrightarrow{\mathrm{id} \odot \eta} Y \odot X \odot Y \xrightarrow{\epsilon \odot \mathrm{id}} U_B \odot Y \cong Y$$

$$X \cong U_A \odot X \xrightarrow{\eta \odot \mathrm{id}} X \odot Y \odot X \xrightarrow{\mathrm{id} \odot \epsilon} X \odot U_B \cong X$$

are the identity maps of  $Y$  and  $X$  respectively.

The map  $\eta$  is called the *coevaluation* and  $\epsilon$  is called the *evaluation*. We say  $(X, Y)$  is a *dual pair*.

If  $M$  is a finitely generated projective right  $R$ -module then  $M$  is a right dualizable 1-cell in  $\mathbf{Mod}$  with dual  $\mathrm{Hom}_R(M, R)$ . The evaluation map

$$\mathrm{Hom}_R(M, R) \otimes_{\mathbb{Z}} M \rightarrow R$$

is the usual evaluation map. This is a map of  $R$ - $R$ -bimodules. Since  $M$  is finitely generated and projective there are elements  $\{m_1, m_2, \dots, m_n\}$  of  $M$  and dual elements  $\{m'_1, m'_2, \dots, m'_n\}$  of  $\mathrm{Hom}_R(M, R)$  so that the coevaluation map

$$\mathbb{Z} \rightarrow M \otimes_R \mathrm{Hom}_R(M, R)$$

is defined by linearly extending  $\eta(1) = \sum m_i \otimes m'_i$ . This is a map of abelian groups.

For some dualizable objects, like modules, it is easy to directly describe the dual, coevaluation and evaluation. For others it is easier to describe the dual in steps. The following lemma describes this process. This lemma is the source of many of the dual pairs we will use.

**Lemma 2.3.** [23, 16.5.1] *If  $X \in \mathcal{B}(A, B)$  and  $Z \in \mathcal{B}(B, C)$  are right dualizable with duals  $Y$  and  $W$  respectively,  $X \odot Z$  is right dualizable with dual  $W \odot Y$ .*

If the coevaluation and evaluation for the dual pair  $(X, Y)$  are  $\eta$  and  $\epsilon$  and those for the dual pair  $(Z, W)$  are  $\chi$  and  $\theta$ , the coevaluation for  $(X \odot Z, W \odot Y)$  is

$$U_A \xrightarrow{\eta} X \odot Y \cong X \odot U_B \odot Y \xrightarrow{\mathrm{id} \odot \chi \odot \mathrm{id}} X \odot Z \odot W \odot Y.$$

The evaluation is

$$W \odot Y \odot X \odot Z \xrightarrow{\mathrm{id} \odot \epsilon \odot \mathrm{id}} W \odot U_B \odot Z \cong W \odot Z \xrightarrow{\theta} U_C.$$

Like the symmetric monoidal case, the trace of a 2-cell is defined using a composite of the coevaluation and evaluation for a dual pair. Unlike that case, the source of the evaluation and target of the coevaluation are not isomorphic. To accommodate this, we need more structure on a bicategory before we can define trace. The additional structure is a shadow.

**Definition 2.4.** [24, 4.4.1] A *shadow* for  $\mathcal{B}$  is a functor

$$\langle\langle - \rangle\rangle: \coprod \mathcal{B}(A, A) \rightarrow \mathcal{T}$$

to a category  $\mathcal{T}$  and a natural isomorphism

$$\langle\langle X \odot Y \rangle\rangle \cong \langle\langle Y \odot X \rangle\rangle$$

for every pair of 1-cells  $X \in \mathcal{B}(A, B)$  and  $Y \in \mathcal{B}(B, A)$  such that the diagrams

$$\begin{array}{ccccc} \langle\langle (X \odot Y) \odot Z \rangle\rangle & \longrightarrow & \langle\langle Z \odot (X \odot Y) \rangle\rangle & \longrightarrow & \langle\langle (Z \odot X) \odot Y \rangle\rangle \\ \downarrow & & & & \uparrow \\ \langle\langle X \odot (Y \odot Z) \rangle\rangle & \longrightarrow & \langle\langle (Y \odot Z) \odot X \rangle\rangle & \longrightarrow & \langle\langle Y \odot (Z \odot X) \rangle\rangle \\ & & \langle\langle Z \odot U_A \rangle\rangle \longrightarrow \langle\langle U_A \odot Z \rangle\rangle \longrightarrow \langle\langle Z \odot U_A \rangle\rangle & & \\ & & \searrow \downarrow \swarrow & & \\ & & \langle\langle Z \rangle\rangle & & \end{array}$$

commute.

Let  $P$  be an  $R$ - $R$ -bimodule. Let  $N(P)$  be the subgroup of  $P$  generated by elements of the form

$$rp - pr$$

for  $p \in P$  and  $r \in R$ . The shadow of  $P$  is  $P/N(P)$ . This defines a shadow on the bicategory of bimodules with the required isomorphisms given by the transposition of elements. Notice the shadow is very similar to the definition of the tensor product of two modules.

**Definition 2.5.** [24, 4.5] Let  $X$  be a dualizable 1-cell in  $\mathcal{B}$  with dual  $Y$  and  $f: Q \odot X \rightarrow X \odot P$  be a 2-cell in  $\mathcal{B}$ . The *trace* of  $f$  is the composite

$$\begin{aligned} \langle\langle Q \rangle\rangle &\cong \langle\langle Q \odot U_A \rangle\rangle \xrightarrow{\text{id} \odot \eta} \langle\langle Q \odot X \odot Y \rangle\rangle \\ &\quad \downarrow f \odot \text{id} \\ \langle\langle X \odot P \odot Y \rangle\rangle &\xrightarrow{\sim} \langle\langle P \odot Y \odot X \rangle\rangle \xrightarrow{\text{id} \odot \epsilon} \langle\langle P \odot U_B \rangle\rangle \cong \langle\langle P \rangle\rangle. \end{aligned}$$

If  $M$  is a finitely generated projective right  $R$ -module and  $f: M \rightarrow M$  is a  $R$ -module homomorphism the trace of  $f$  is the trace defined by Stallings in [29].

Suppose  $\mathcal{B}$  is a bicategory with a shadow  $\langle\langle - \rangle\rangle$  that takes values in a category  $\mathcal{T}$  and  $\mathcal{B}'$  is a bicategory with a shadow  $\langle\langle - \rangle\rangle'$  that takes values in a category  $\mathcal{T}'$ . A *shadow functor*

$$F: \mathcal{B} \rightarrow \mathcal{B}'$$

is a functor of bicategories  $F: \mathcal{B} \rightarrow \mathcal{B}'$ , a functor  $F_t: \mathcal{T} \rightarrow \mathcal{T}'$ , and a natural transformation

$$\psi: \langle\langle F(-) \rangle\rangle' \rightarrow F_t(\langle\langle - \rangle\rangle)$$

such that

$$\begin{array}{ccc} \langle\langle FX \odot FY \rangle\rangle' & \longrightarrow & \langle\langle FY \odot F(X) \rangle\rangle' \\ \downarrow & & \downarrow \\ \langle\langle F(X \odot Y) \rangle\rangle' & & \langle\langle F(Y \odot X) \rangle\rangle' \\ \downarrow \psi & & \downarrow \psi \\ F_t(\langle\langle X \odot Y \rangle\rangle) & \longrightarrow & F_t(\langle\langle Y \odot X \rangle\rangle) \end{array}$$

commutes for each pair of compatible 1-cells  $X$  and  $Y$ .

**Proposition 2.6.** [24, 4.5.7] Let  $F$  be a shadow functor,  $X$  be a dualizable 1-cell in  $\mathcal{B}$  with dual  $Y$ , and

$$f: Q \odot X \rightarrow X \odot P$$

be a 2-cell. If

$$\begin{aligned} F(X) \odot F(Y) &\rightarrow F(X \odot Y), \\ F(X) \odot F(P) &\rightarrow F(X \odot P), \end{aligned}$$

and

$$U_{F(B)} \rightarrow F(U_B)$$

are isomorphisms and  $\hat{f}$  is the composite

$$FQ \odot FX \xrightarrow{\phi} F(Q \odot X) \xrightarrow{f} F(X \odot P) \xrightarrow{\phi^{-1}} FX \odot FP.$$



then

$$\begin{array}{ccc} \langle\langle FQ \rangle\rangle & \xrightarrow{\text{tr}(\hat{f})} & \langle\langle FP \rangle\rangle \\ \downarrow \psi & & \downarrow \psi \\ F\langle\langle Q \rangle\rangle & \xrightarrow{F(\text{tr}(f))} & F\langle\langle P \rangle\rangle \end{array}$$

commutes.

This proposition will be an important part of the comparisons of fixed point invariants.

### 3. THE BICATEGORY OF ENRICHED DISTRIBUTORS

In this section and the next we will describe the bicategories we will use to define equivariant fixed point invariants. These bicategories are generalizations of the bicategory of rings, bimodules and homomorphisms. The first is the bicategory of categories, distributors, and natural transformations enriched in a symmetric monoidal category. We define this bicategory and prove some basic properties in this section. The second is the bicategory of monoids, bimodules, and homomorphisms in a bicategory. We describe this bicategory in Section 4. Both of these bicategories are described in [24, 26]. There are many examples of how these bicategories are used to define classical fixed point invariants in [24].

We will use the bicategory of distributors to define the invariants in Sections 6, 7, 10, and 11. We will give examples of dualizable 1-cells in this bicategory in Sections 5 and 9.

Let  $\mathcal{V}$  be a symmetric monoidal category with product  $\otimes$  and unit  $S$ . A category  $\mathcal{A}$  is *enriched* in  $\mathcal{V}$  if for pairs of objects  $a$  and  $b$  in  $\mathcal{A}$ ,  $\mathcal{A}(a, b)$  is an object of  $\mathcal{V}$  and for objects  $a, b$ , and  $c$  in  $\mathcal{A}$ , the composition

$$\mathcal{A}(a, b) \otimes \mathcal{A}(b, c) \rightarrow \mathcal{A}(a, c),$$

is a morphism in  $\mathcal{V}$ .

If  $\mathcal{A}$  and  $\mathcal{B}$  are enriched categories, the objects of the category

$$\mathcal{A} \otimes \mathcal{B}$$

are pairs  $(a, b)$  where  $a \in \text{ob}\mathcal{A}$  and  $b \in \text{ob}\mathcal{B}$ . If  $a, a' \in \text{ob}\mathcal{A}$  and  $b, b' \in \text{ob}\mathcal{B}$ , then

$$(\mathcal{A} \otimes \mathcal{B})((a, b), (a', b')) = (\mathcal{A}(a, a')) \otimes (\mathcal{B}(b, b')).$$

Define an enriched category  $S_{\mathcal{V}}$  with one object  $*$  and  $S_{\mathcal{V}}(*, *) = S$ . Then  $\mathcal{A} \otimes S_{\mathcal{V}}$  is isomorphic to  $\mathcal{A}$ .

A functor  $\mathcal{X}: \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathcal{V}$  is an *enriched distributor* if the composition map

$$\mathcal{A}(a, a') \otimes \mathcal{X}(a, b') \otimes \mathcal{B}(b, b') \rightarrow \mathcal{X}(a', b)$$

is a map in  $\mathcal{V}$  for all objects  $a$  and  $a'$  of  $\mathcal{A}$  and  $b$  and  $b'$  of  $\mathcal{B}$ . Distributors of this form are also called  $\mathcal{A}$ - $\mathcal{B}$ -bimodules. A distributor

$$\mathcal{X}: \mathcal{A} \cong \mathcal{A} \otimes S_{\mathcal{V}}^{\text{op}} \rightarrow \mathcal{V}$$

is a left  $\mathcal{A}$ -module and a distributor

$$\mathcal{X}: \mathcal{B}^{\text{op}} \cong S_{\mathcal{V}} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathcal{V}$$

is a right  $\mathcal{B}$ -module.

If  $\mathcal{V}$  is the category of abelian groups and  $\mathcal{A}$  and  $\mathcal{B}$  are categories with one object,  $\mathcal{A}$  and  $\mathcal{B}$  are rings. A distributor

$$\mathcal{X}: \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathcal{V}$$

is a bimodule in the usual sense.

For a category  $\mathcal{A}$  define a distributor

$$U_{\mathcal{A}}: \mathcal{A} \otimes \mathcal{A}^{\text{op}} \rightarrow \mathcal{V}$$

by  $U_{\mathcal{A}}(a, a') = \mathcal{A}(a', a)$ . When there is no ambiguity, we will use  $\mathcal{A}$  to denote the distributor  $U_{\mathcal{A}}$ . If  $F: \mathcal{A} \rightarrow \mathcal{C}$  is a functor, define a distributor

$$\mathcal{C}^F: \mathcal{C} \otimes \mathcal{A}^{\text{op}} \rightarrow \mathcal{V}$$

by

$$\mathcal{C}^F(c, a) = \mathcal{C}(F(a), c).$$

An *enriched natural transformation*  $\eta: \mathcal{X} \rightarrow \mathcal{Y}$  between two distributors  $\mathcal{X}: \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathcal{V}$  and  $\mathcal{Y}: \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathcal{V}$  is a natural transformation where the maps

$$\eta_{a,b}: \mathcal{X}(a, b) \rightarrow \mathcal{Y}(a, b)$$

are maps in  $\mathcal{V}$  for each  $a \in \text{ob}\mathcal{A}$  and  $b \in \text{ob}\mathcal{B}$ .

If  $\mathcal{X}: \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathcal{V}$  and  $\mathcal{Y}: \mathcal{B} \otimes \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$  are two distributors, the distributor  $\mathcal{X} \odot \mathcal{Y}: \mathcal{A} \otimes \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$  is the coequalizer of the diagram

$$\coprod_{b, b' \in \text{ob}\mathcal{B}} \mathcal{X}(a, b) \otimes \mathcal{B}(b', b) \otimes \mathcal{Y}(b', c) \rightrightarrows \coprod_{b \in \text{ob}\mathcal{B}} \mathcal{X}(a, b) \otimes \mathcal{Y}(b, c)$$

for all  $a \in \text{ob}\mathcal{A}$  and  $c \in \text{ob}\mathcal{C}$ . This generalizes the usual tensor product of modules. For  $\mathcal{X}$  as above,  $\mathcal{X} \odot U_{\mathcal{B}} \cong \mathcal{X} \cong U_{\mathcal{A}} \odot \mathcal{X}$ . If  $F: \mathcal{C} \rightarrow \mathcal{B}$  is a functor,  $\mathcal{X} \odot \mathcal{C}^F$  is equivalent to the distributor

$$\mathcal{A} \otimes \mathcal{C}^{\text{op}} \xrightarrow{\text{id} \otimes F} \mathcal{A} \otimes \mathcal{B}^{\text{op}} \xrightarrow{\mathcal{X}} \mathcal{V}.$$

Let  $\mathcal{E}_{\mathcal{V}}$  denote the bicategory with 0-cells  $\mathcal{V}$ -enriched categories, 1-cells  $\mathcal{V}$ -enriched distributors, and 2-cells  $\mathcal{V}$ -enriched natural transformations. See [14] for more about enriched categories.

If  $\mathcal{Z}: \mathcal{A} \otimes \mathcal{A}^{\text{op}} \rightarrow \mathcal{V}$  is a distributor, the shadow of  $\mathcal{Z}$  is the coequalizer of

$$\coprod_{a, a' \in \text{ob}(\mathcal{A})} \mathcal{A}(a, a') \otimes \mathcal{Z}(a, a') \rightrightarrows \coprod_{a \in \text{ob}(\mathcal{A})} \mathcal{Z}(a, a).$$

The symmetry isomorphism in  $\mathcal{V}$  defines the map

$$\langle\langle \mathcal{X} \odot \mathcal{Y} \rangle\rangle \rightarrow \langle\langle \mathcal{Y} \odot \mathcal{X} \rangle\rangle.$$

This generalizes the shadow of a bimodule and is compatible with the bicategory composition of distributors.

In [24] we used the following lemma to verify certain distributors were dualizable.

**Lemma 3.1.** [24, 9.2.6] *If  $\mathcal{B}$  is a connected groupoid, a distributor*

$$\mathcal{X}: \mathcal{B}^{\text{op}} \rightarrow \mathcal{V}$$

*is dualizable if and only if  $\mathcal{X}(b)$  is dualizable as a distributor*

$$\mathcal{X}(b): \mathcal{B}(b, b)^{\text{op}} \rightarrow \mathcal{V}$$

*for any  $b \in \text{ob}\mathcal{B}$ .*

The categories we use to describe equivariant fixed point invariants will usually not be groupoids. However, there is a generalization of this characterization that we will use to verify the distributors we consider are dualizable. Both the characterization above and the one we describe now are related to the general description in [2, 7.9.2].

**Definition 3.2.** [20, II.9.2] A category  $\mathcal{A}$  is an *EI-category* if all endomorphisms are isomorphisms.

In an EI-category there is a partial order on the set of objects:  $x < y$  if  $\mathcal{A}(x, y) \neq \emptyset$ . The following three categories are EI-categories.

**Definition 3.3.** [6, I.10.1] The objects of the *orbit category* of a finite group  $G$ ,  $\mathcal{O}_G$ , are the subgroups of  $G$ . The morphisms from a subgroup  $H$  to a subgroup  $K$  are the  $G$ -maps

$$G/H \rightarrow G/K.$$

If  $a$  is an element of  $G$  such that  $aHa^{-1} \subset K$ , there is a  $G$ -map

$$R_a: G/H \rightarrow G/K$$

defined by  $R_a(gH) = gaK$ . All  $G$ -maps  $G/H \rightarrow G/K$  are of this form.

**Definition 3.4.** [6, I.10.3] The objects of the *equivariant component category* of a  $G$ -space  $X$ ,  $\Pi_0(G, X)$ , are  $G$ -maps  $x(H): G/H \rightarrow X$ . A morphism from  $x(H): G/H \rightarrow X$  to  $y(K): G/K \rightarrow X$  is a  $G$  map

$$R_a: G/H \rightarrow G/K$$

such that  $y(K) \circ R_a$  and  $x(H)$  are  $G$ -homotopic.

**Definition 3.5.** [6, I.10.7] The objects of the *equivariant fundamental category* of a  $G$ -space  $X$ ,  $\Pi(G, X)$ , are the  $G$ -maps  $x(H): G/H \rightarrow X$ . A morphism from  $x(H)$  to  $y(K)$  is a  $G$ -map

$$R_a: G/H \rightarrow G/K$$

and a homotopy class of  $G$ -maps

$$w(H): G/H \times I \rightarrow X$$

relative to  $G/H \times \partial I$  such that  $w(H)(-, 0) = x(H)$  and  $w(H)(-, 1) = y(K) \circ R_a$ .

In general,  $\Pi_0(G, X)$  and  $\Pi(G, X)$  are not groupoids.

Let  $\mathbf{Ch}_R$  be the category of chain complexes of modules over a commutative ring  $R$ . Let  $\mathcal{A}$  be an EI-category enriched in abelian groups.

**Definition 3.6.** [25, 3.5] A functor  $\mathcal{X}: \mathcal{A} \rightarrow \mathbf{Ch}_R$  is *supported on isomorphisms* if  $\mathcal{X}(f)$  is the zero map if  $f$  is not an isomorphism.

If  $\mathcal{X}$  is supported on isomorphisms it only ‘sees’ a disjoint collection of groupoids rather than the entire category  $\mathcal{A}$ . Let  $B(\mathcal{A})$  denote a choice of representative for each isomorphism class of objects in  $\mathcal{A}$ .

**Lemma 3.7.** [25, 3.6] *Let  $\mathcal{X}: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Ch}_R$  and  $\mathcal{Y}: \mathcal{A} \rightarrow \mathbf{Ch}_R$  be supported on isomorphisms. Then*

$$\mathcal{X} \odot \mathcal{Y} \cong \bigoplus_{c \in B(\mathcal{A})} \mathcal{X}(c) \otimes_{\mathcal{A}(c, c)} \mathcal{Y}(c).$$

The idea of the proof is to use Definition 3.6 to show that

$$\bigoplus_{c \in B(\mathcal{A})} \mathcal{X}(c) \otimes_{\mathcal{A}(c,c)} \mathcal{Y}(c)$$

satisfies the universal property that defines  $\mathcal{X} \odot \mathcal{Y}$ .

**Corollary 3.8.** [25, 3.7] *Let  $\mathcal{X}$  and  $\mathcal{Y}$  satisfy the conditions of Lemma 3.7. If  $\mathcal{X}(c)$  is dualizable as an  $\mathcal{A}(c,c)$ -module with dual  $\mathcal{Y}(c)$  for each  $c \in B(\mathcal{A})$  then  $\mathcal{X}$  is dualizable with dual  $\mathcal{Y}$ .*

The idea of this proof is to use Lemma 3.7 and the coevaluation and evaluation maps for each  $\mathcal{X}(c)$  to build a coevaluation and evaluation for  $\mathcal{X}$ .

There are similar restrictions on the shadows that can arise from EI-categories.

**Lemma 3.9.** *Suppose  $F: \mathcal{A} \rightarrow \mathcal{A}$  is a functor such that for objects  $a$  and  $b$  in  $B(\mathcal{A})$  with  $a \neq b$  at least one of  $\mathcal{A}(F(a), b)$  or  $\mathcal{A}(b, a)$  is empty. Then*

$$\langle\langle \mathcal{A}^F \rangle\rangle \cong \bigoplus_{b \in B(\mathcal{A})} \langle\langle \mathcal{A}(F(b), b) \rangle\rangle.$$

The hypothesis on the functor  $F$  implies the coequalizer that defines the shadow of  $\mathcal{A}^F$  reduces to a coequalizer for each  $b \in B(\mathcal{A})$ .

Some of our invariants will be defined using distributors enriched in the category of based spaces,  $\mathbf{Top}_*$ . For this category we need to make a few modifications to the definition of the bicategory composition, the shadow and dualizable objects. These changes ensure the spaces all have the correct homotopy type.

If  $\mathcal{X}: \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathbf{Top}_*$  and  $\mathcal{Y}: \mathcal{B} \otimes \mathcal{C}^{\text{op}} \rightarrow \mathbf{Top}_*$  are two distributors

$$\mathcal{X} \odot \mathcal{Y}: \mathcal{A} \otimes \mathcal{C}^{\text{op}} \rightarrow \mathbf{Top}_*$$

is the bar resolution  $B(\mathcal{X}, \mathcal{B}, \mathcal{Y})$ . For an object  $a$  in  $\mathcal{A}$  and an object  $c$  in  $\mathcal{C}$ , the space  $B(\mathcal{X}, \mathcal{B}, \mathcal{Y})(a, c)$  is the geometric realization of the simplicial topological space with  $n$ -simplices

$$\coprod_{b_0, b_1, b_2, \dots, b_n \in \text{ob } \mathcal{B}} \mathcal{X}(a, b_0) \wedge \mathcal{A}(b_1, b_0) \wedge \dots \wedge \mathcal{A}(b_n, b_{n-1}) \wedge \mathcal{Y}(b_n, c).$$

Note that  $B(\mathcal{X}, \mathcal{A}, \mathcal{A})$  is equivalent to  $\mathcal{X}$ .

If  $\mathcal{Z}: \mathcal{A} \otimes \mathcal{A}^{\text{op}} \rightarrow \mathbf{Top}_*$  is a distributor, the shadow of  $\mathcal{Z}$  is the cyclic bar resolution. The cyclic bar resolution is the geometric realization of the simplicial space with  $n$ -simplices

$$\coprod_{a_0, a_1, a_2, \dots, a_n \in \text{ob } \mathcal{A}} \mathcal{Z}(a_n, a_0) \wedge \mathcal{A}(a_1, a_0) \wedge \dots \wedge \mathcal{A}(a_n, a_{n-1}).$$

See [24, 3.1] for more details.

In this bicategory we define dualizable 1-cells using homotopy classes of maps.

**Definition 3.10.** A functor  $\mathcal{X}: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Top}_*$  is  $n$ -dualizable if there is a functor  $\mathcal{Y}: \mathcal{A} \rightarrow \mathbf{Top}_*$ , a map  $\eta: S^n \rightarrow B(\mathcal{X}, \mathcal{A}, \mathcal{Y})$ , and an  $\mathcal{A}$ - $\mathcal{A}$ -equivariant map  $\epsilon: \mathcal{Y} \wedge \mathcal{X} \rightarrow S^n \wedge \mathcal{A}$  such that

$$\begin{array}{ccc} S^n \wedge \mathcal{X} & \xrightarrow{\eta \wedge \text{id}} & B(\mathcal{X}, \mathcal{A}, \mathcal{Y}) \wedge \mathcal{X} \\ & \searrow \gamma & \downarrow \text{id} \wedge \epsilon \\ & & \mathcal{X} \wedge S^n \end{array} \quad \begin{array}{ccc} \mathcal{Y} \wedge S^n & \xrightarrow{\text{id} \wedge \eta} & \mathcal{Y} \wedge B(\mathcal{X}, \mathcal{A}, \mathcal{Y}) \\ & \searrow (\sigma \wedge \text{id}) \gamma & \downarrow \epsilon \wedge \text{id} \\ & & S^n \wedge \mathcal{Y} \end{array}$$

commute up to  $\mathcal{A}$ -equivariant homotopy.

If  $\mathcal{A}$  is a category with one object  $*$  and the morphisms of  $\mathcal{A}$  are a discrete, finite set, a functor  $\mathcal{X}: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Top}_*$  is a space  $X$  with an action by the group  $\mathcal{A}(*, *)$ . Then the definition above reduces to the following statement: If  $X$  is an  $\mathcal{A}(*, *)$ -space then  $X$  is dualizable if there is an  $\mathcal{A}(*, *)$ -space  $Y$ , a map

$$\eta: S^n \rightarrow X \wedge_{\mathcal{A}(*, *)} Y$$

and an  $\mathcal{A}(*, *)$ - $\mathcal{A}(*, *)$ -equivariant map

$$Y \wedge X \rightarrow \vee_{\mathcal{A}(*, *)} S^n$$

such that the triangle diagrams above commute. This type of duality is called *Ranicki duality*.

In general this is different from the duality of Definition 1.4, but in one important case these duals coincide. Recall that  $G$  is a finite group.

**Theorem 3.11.** [1, 8.6] *Let  $X$  be a free  $G$ -space. If  $X$  is Ranicki dualizable and equivariantly dualizable then the Ranicki dual is equivalent to the equivariant dual.*

We will use this theorem and its proof to connect different descriptions of the Lefschetz number and Reidemeister trace.

In the topological examples we will consider here a condition like the condition in Definition 3.6 will be satisfied. We will use this perspective to construct dual pairs following the approach of Corollary 3.8.

#### 4. THE BICATEGORY OF BIMODULES

The bicategory of bimodules is another generalization of the bicategory  $\mathbf{Mod}$ . We will use this bicategory to define the invariant in Section 12.

A *monoid* in a symmetric monoidal category  $\mathcal{V}$  is an object  $E$  of  $\mathcal{V}$  and two morphisms

$$\mu: E \otimes E \rightarrow E$$

and

$$\kappa: S \rightarrow E$$

that satisfy unit and associativity axioms. If  $E$  and  $E'$  are monoids in  $\mathcal{V}$ , an  *$E$ - $E'$ -bimodule* is an object  $M$  of  $\mathcal{V}$  and two morphisms

$$E \otimes M \rightarrow M$$

and

$$M \otimes E' \rightarrow M$$

that satisfy compatibility axioms. If  $M$  and  $N$  are  $E$ - $E'$ -bimodules, a *bimodule homomorphism*

$$M \rightarrow N$$

is a morphism in  $\mathcal{V}$  that respects the action of  $E$  and  $E'$ . The monoids, bimodules, and homomorphism in a symmetric monoidal category  $\mathcal{V}$  define a bicategory. In the case  $\mathcal{V}$  is the category of abelian groups, this bicategory is the bicategory  $\mathbf{Mod}$ .

We can generalize this to the bicategory of monoids, bimodules, and homomorphisms in a bicategory rather than a symmetric monoidal category. The general construction is described in [24, 26]. We will only consider the case of monoids and bimodules in the bicategory of parametrized  $G$ -spaces here.

The definition of the bicategory of parametrized spaces can be found in Section 16. That section also includes theorems that we will use later in this paper.

**Definition 4.1.** A *monoid* is an ex- $G$ -space  $E$  over  $B \times B$  and parametrized equivariant maps

$$\mu: E \boxtimes E \rightarrow E$$

and

$$\kappa: U_B := (B \amalg (B \times B)) \rightarrow E$$

that are associative and unital.

The  $\boxtimes$  product is defined in Section 16. It is the bicategory composition in the bicategory of parametrized  $G$ -spaces. The section of  $U_B$  is the inclusion of  $B \times B$  and the projection is the disjoint union of the diagonal map and the identity map. The space  $U_B$  is a unit for  $\boxtimes$ .

If  $B$  is a  $G$ -space, the ex- $G$ -space  $U_B$  is a monoid. The map  $\mu$  the unit isomorphism and  $\kappa$  the identity map.

Let  $e$  be the group with one element and  $\pi$  be a discrete group. Then  $\pi_+ := \pi \amalg *$  is an ex- $e$ -space over  $*$ . The group multiplication defines a map  $\mu$  and  $\kappa$  is the inclusion of  $S^0$  into  $\pi_+$  as the identity element and the disjoint base point.

Let  $X$  be a  $G$ -space. The space of Moore paths in  $X$ ,

$$\mathcal{P}(X) := \{(\gamma, u) \in X^{[0, \infty)} \times [0, \infty) \mid \gamma(v) = \gamma(u) \text{ for all } v \geq u\},$$

is a  $G$  space where  $G$  acts trivially on  $[0, \infty)$ . The evaluation maps

$$s: \mathcal{P}(X) \rightarrow X \text{ and } t: \mathcal{P}(X) \rightarrow X,$$

defined by  $s(\gamma, u) = \gamma(0)$  and  $t(\gamma, u) = \gamma(u)$ , are equivariant. Let

$$(\mathcal{P}(X), t \times s)_+$$

be the ex- $G$ -space with total space  $\mathcal{P}(X) \amalg (X \times X)$ , projection  $(t \times s) \amalg \text{id}$  and section the inclusion of  $X \times X$ . Composition of paths makes  $\mathcal{P}(X)$  a monoid in  $G\text{Ex}$ .

**Definition 4.2.** A *right  $E$ -module* is an ex- $G$ -space  $M$  over  $B$  and a parametrized equivariant map

$$m: M \boxtimes E \rightarrow M$$

which satisfies the usual module axioms with respect to the maps  $\mu$  and  $\kappa$ .

Left modules and bimodules are defined similarly.

If  $X$  is a  $\pi$  space then  $X_+$  is a module over  $\pi_+$  in the bicategory of parametrized  $e$ -spaces. The map

$$m: X_+ \wedge \pi_+ = (X \times \pi)_+ \rightarrow X_+$$

is the action of  $\pi$  on  $X$ . If  $E$  is an ex-space over  $B$  and  $E$  has a fiberwise right action by a group  $\pi$  we can think of  $E$  as a  $U_B\text{-}\pi_+$ -bimodule.

The path space  $\mathcal{P}(X)$  defines several modules by neglect of structure. Using only composition on the right  $(\mathcal{P}(X), t \times s)_+$  is a space over  $X \times X$  and is a  $U_X\text{-}\mathcal{P}(X)$ -module. Let

$$R(\mathcal{P}(X), t \times s)$$

denote this module. Similarly, let

$$L(\mathcal{P}(X), t \times s)$$

denote the associated  $\mathcal{P}(X)\text{-}U_X$ -module given by using only left composition. Let  $(\mathcal{P}(X), s)_+$  be the space over  $X$  with projection  $s$  and a disjoint section  $X$ . This is a right  $\mathcal{P}(X)$ -module. Similarly,  $(\mathcal{P}(X), t)_+$  is a left  $\mathcal{P}(X)$ -module.

**Definition 4.3.** If  $M$  is a right  $E$ -module and  $N$  is a left  $E$ -module  $M \odot N$  is the bar resolution  $B(M, E, N)$ .

The bar resolution  $B(M, E, N)$  is the geometric realization of the simplicial space with  $n$  simplices

$$M \boxtimes E^n \boxtimes N.$$

The face and degeneracy maps are defined using the maps  $\mu, \kappa, m_M$  and  $m_N$ . For  $M$  and  $N$  as above,  $N \odot M$  is the ex- $G$ -space  $N \bar{\wedge} M$  over  $B \times B$  with actions of  $E$  on the left and right. The composition  $\bar{\wedge}$  is the external product. It is defined in [23, 2.5] and also described in Section 16.

Let  $V$  be a  $G$ -representation.

**Definition 4.4.** A right  $E$ -module  $M$  is *right  $V$ -dualizable* if there is a left  $E$ -module  $N$ , a  $G$ -equivariant map

$$\eta: S^V \rightarrow M \odot N,$$

and a  $E$ - $E$ -bimodule homomorphism

$$\epsilon: N \odot M \rightarrow S^V \wedge E$$

such that

$$\begin{array}{ccc} S^V \wedge M & \xrightarrow{\eta \wedge \text{id}} & M \odot N \odot M \\ & \searrow \gamma & \downarrow \text{id} \wedge \epsilon \\ & & M \wedge S^V \end{array} \quad \begin{array}{ccc} N \wedge S^V & \xrightarrow{\text{id} \wedge \eta} & N \odot M \odot N \\ & \searrow (\sigma \wedge \text{id}) \gamma & \downarrow \epsilon \wedge \text{id} \\ & & S^V \wedge N \end{array}$$

commute up to  $E$ -equivariant homotopy.

The following lemma is the basis of many of the dual pairs we consider.

**Lemma 4.5.** *If  $X$  is a  $G$ -space then*

$$(R(\mathcal{P}(X), t \times s)_+, L(\mathcal{P}(X), t \times s)_+)$$

*is a dual pair.*

*Proof.* The coevaluation map

$$U_X \rightarrow (\mathcal{P}(X), t \times s)_+ \cong R((\mathcal{P}(X), t \times s)_+) \odot L((\mathcal{P}(X), t \times s)_+)$$

is the inclusion. The evaluation map

$$\begin{array}{c} L((\mathcal{P}(X), t \times s)_+) \odot R((\mathcal{P}(X), t \times s)_+) \\ \downarrow \cong \\ L((\mathcal{P}(X), t \times s)_+) \bar{\wedge} R((\mathcal{P}(X), t \times s)_+) \longrightarrow (\mathcal{P}(X), t \times s)_+ \end{array}$$

is composition of paths. The monoid axioms imply that these maps are maps of modules and that the required diagrams commute.  $\square$

**Definition 4.6.** If  $P$  is an  $E$ - $E$ -bimodule the *shadow* of  $P$ ,  $\langle\langle P \rangle\rangle$ , is the cyclic bar resolution  $C(E, P)$ .

The cyclic bar resolution is the geometric realization of the simplicial space with  $n$ -simplices

$$\pi_! \Delta^*(E^n \boxtimes P).$$

The functors  $\pi_!$  and  $\Delta^*$  are defined in Section 16. The face and degeneracy maps are defined using  $\kappa$ ,  $\mu$ , and  $m$ .

## 5. THE EQUIVARIANT COMPONENT SPACE

In the first section we defined the local Lefschetz number. In Section 7 we will define the global Lefschetz number. The equivariant Lefschetz fixed point theorem is a consequence of the identification these two invariants.

We will make this identification in two steps. The first step, which we complete in the next section, is the identification of the local Lefschetz number with a third invariant, the geometric Lefschetz number. The geometric Lefschetz number is a trace in the bicategory of categories, bimodules, and natural transformations enriched in topological spaces. In this section we define the distributor used to describe the geometric Reidemeister trace. We also prove this distributor is dualizable. We complete the identification of the local and global Lefschetz numbers in Section 7 with the identification of the geometric and global Lefschetz numbers.

**Definition 5.1.** The *equivariant component space* of  $X$ ,  $\overline{G|X}$ , is the functor

$$\Pi_0(G, X)^{\text{op}} \rightarrow \mathbf{Top}_*$$

defined by  $\overline{G|X}(x(H)) = X^H(x)/X^{>H}(x)$ . The action of the morphisms in  $\Pi_0(G, X)$  is induced by the group action.

In order to think of  $\overline{G|X}$  as an enriched distributor we need to think of  $\Pi_0(G, X)$  as a category enriched in based topological spaces. For this we think of each of the hom sets as a discrete space and add a disjoint base point to each.

Proposition 0.3 implies the space  $X^H(x)/X^{>H}(x)$  is  $WH$ -equivariantly equivalent to the mapping cone of the inclusion  $X^{>H}(x) \subset X^H(x)$ .

**Proposition 5.2.** *If  $X$  is a compact  $G$ -ENR then  $\overline{G|X}$  is dualizable.*

Before we prove this proposition we prove two preliminary lemmas.

**Lemma 5.3.** *If  $X$  is a compact  $G$ -ENR then  $X^H/WH$  is an ENR.*

*Proof.* If  $X$  is a  $G$ -ENR, then  $X^H$  and  $X^{>H}$  are  $WH$ -ENR's. See [30, II.6.7] for a proof.

If  $X$  is a  $G$ -ENR, then by [5, 5.2.5]  $X/G$  is an ENR. So  $X^H/WH$  and  $X^{>H}/WH$  are ENR's.  $\square$

**Lemma 5.4.** *Suppose  $L$  is a discrete group,  $Y$  is a based  $L$ -space, and the action of  $L$  on  $Y$  is free away from the base point. If  $Y/L$  is a compact ENR then  $Y$  is Ranicki dualizable.*

If the group  $L$  is finite this is a consequence of Theorem 3.11 and the dualizability of compact ENR's in the equivariant stable homotopy category. If  $Y$  is a finite  $L$ -CW complex this result is [28, 3.5]. We include a proof here since we will need descriptions of the coevaluation and evaluation maps.

The proof of this lemma is very similar to the corresponding proofs [8, 3.1] and [23, 18.8].



*Proof.* Embed  $Y/L$  in a representation  $V$ . Let  $N$  be an open neighborhood of  $Y/L$  with a retraction  $r: N \rightarrow Y/L$ . Let

$$C_{Y/L}(N \setminus *, N \setminus Y/L) := (N \setminus *) \times \{0\} \cup (N \setminus Y/L) \times I \cup Y/L \times \{1\}.$$

This is a space over  $Y/L$  with projection  $p$  given by the retraction  $r$ . If  $x \in Y$ , let  $\bar{x}$  be the image of  $x$  in  $Y/L$ . Let  $DY$  be the space

$$\{(x, v) \in Y \times C_{Y/L}(N \setminus *, N \setminus (Y/L)) \mid \bar{x} = p(v)\} / \sim$$

where  $(x, v) \sim (y, u)$  if  $v$  and  $u$  are in  $Y/L \times \{1\}$ .

Let  $B$  be a ball in  $V$  that contains  $Y/L$ . To simplify notation in the following diagrams we use  $(E, F)$  to denote  $E \cup CF$  for  $F \subset E$ .

Let  $\eta$  be the composite

$$\begin{array}{ccccc} & & & & (Y, *) \wedge_L DY \\ & & & & \downarrow \\ & & & & (Y \times_L Y \times_{Y/L} (N \setminus *), C_3) \\ & & & \uparrow & \\ (V, V \setminus 0) & & & & (Y \times_L Y \times_{Y/L} (N \setminus *), C_2) \\ \uparrow \cup & \longrightarrow & (V, (V \setminus (Y/L)) \cup *) & & \uparrow \\ (V, V \setminus B) & & \uparrow \cup & & \uparrow R \\ & & (N \setminus *, N \setminus (Y/L)) & \xrightarrow{\Delta} & (N \times_{Y/L} (N \setminus *), C_1) \end{array}$$

where

$$\begin{aligned} C_1 &= N \times_{Y/L} (N \setminus (Y/L)), \\ C_2 &= Y \times_L Y \times_{Y/L} (N \setminus (Y/L)), \\ C_3 &= * \times_L Y \times_{Y/L} (N \setminus *) \cup Y \times_L Y \times_{Y/L} (N \setminus (Y/L)). \end{aligned}$$

The first three maps are inclusion maps. The first map is an equivalence by inspection. Homotopy excision implies the third map is an equivalence. The map  $R$  is defined by  $R(n, v) = (x, x, v)$  for some  $x \in Y$  such that  $\bar{x} = r(n)$ . The second to last map is the inclusion. The last map is the addition on cone coordinates.

Since  $Y/L$  is compact, there is a real number  $\epsilon$  such that every disk of radius  $\epsilon$  centered at a point of  $Y/L$  in  $V$  is contained in  $N$ . Replace  $N$  by the union of these disks. Let  $\Gamma := \{(n, \bar{x}) \in (N \setminus *) \times Y/L \mid \|r(n) - \bar{x}\| < \epsilon\}$  and

$$\Gamma_{\epsilon/2} := \{(n, \bar{x}) \in (N \setminus *) \times Y/L \mid \|r(n) - \bar{x}\| < \epsilon/2\}.$$

If  $(n, \bar{x}) \in \Gamma$  then  $(1-t)r(n) + t\bar{x}$  is in  $N$  for all  $t \in I$ . Define a path  $\omega_{(n, \bar{x})}$  in  $Y/L$  by

$$\omega_{(n, \bar{x})}(t) = r((1-t)r(n) + t\bar{x}).$$

Let

$$\begin{aligned} \Gamma_* &:= \{(n, \bar{x}) \in \Gamma \mid \omega_{(n, \bar{x})}(t) \neq * \text{ for all } t\}, \\ \tilde{\Gamma} &:= \{(n, x) \in (N \setminus *) \times Y \mid (n, \bar{x}) \in \Gamma_*\}, \end{aligned}$$

and

$$\tilde{\Gamma}_{\epsilon/2} := \{(n, x) \in \tilde{\Gamma} \mid (n, \bar{x}) \in \Gamma_{\epsilon/2}\}.$$

Note that for  $(n, x) \in \tilde{\Gamma}$ , the path  $\omega_{(n, \bar{x})}$  lifts uniquely to a path  $\tilde{\omega}_{(n, \bar{x})}$  in  $Y$  such that  $\tilde{\omega}_{(n, \bar{x})}(1) = x$ .

Let  $\epsilon$  be the composite

$$\begin{array}{ccc}
 DY \wedge (Y, *) & & \\
 \downarrow & & \\
 (Y \times_{Y/L} (N \setminus *) \times Y, D_1) & \longrightarrow & (Y \times_{Y/L} (N \setminus *) \times Y, D_2) \\
 & & \uparrow \cup \\
 & & (Y \times_{Y/L} \tilde{\Gamma}, D_3) \xrightarrow{S} L_+ \wedge (V, V \setminus 0)
 \end{array}$$

where

$$\begin{aligned}
 D_1 &= (Y \times_{Y/L} (N \setminus (Y/L)) \times Y) \cup (Y \times_{Y/L} (N \setminus *) \times *) , \\
 D_2 &= D_1 \cup (Y \times_{Y/L} ((N \times Y) \setminus \tilde{\Gamma}_{\epsilon/2})), \\
 D_3 &= (Y \times_{Y/L} \tilde{\Gamma}) \cap D_2.
 \end{aligned}$$

The first map is the addition on cone coordinates. The second and third maps are inclusions. Homotopy excision implies the third map is an equivalence.

If  $(n, y) \in \tilde{\Gamma}$  there is a unique path  $\tilde{\omega}_{(n, \bar{y})}$  in  $Y$  that ends at  $y$  and projects to  $\omega_{(n, \bar{y})}$ . If  $(x, n, y) \in Y \times_{Y/L} \tilde{\Gamma}$  then  $\tilde{\omega}_{(n, \bar{y})}(0) = xg_{x, y}$  for some unique  $g_{x, y} \in L$ . Let  $S(x, n, y) = (g_{x, y}, n - \bar{y})$ .

To see the composite

$$(V, V \setminus 0) \wedge (Y, *) \xrightarrow{1 \wedge \eta} (Y, *) \wedge_L DY \wedge (Y, *) \xrightarrow{\epsilon \wedge 1} (Y, *) \wedge_L (L_+ \wedge (V, V \setminus 0)) \cong (Y, *) \wedge (V, V \setminus 0)$$

is homotopic to the identity it is enough to see the map  $\tilde{\Gamma} \rightarrow Y \times V$  defined by  $(n, y) \mapsto (xg_{x, y}, n - \bar{y})$ , where  $\bar{x} = n$ , is homotopic to the map  $(n, y) \mapsto (y, n)$ . A homotopy demonstrating this is

$$H(n, y, t) = (\tilde{\omega}_{(n, \bar{y})}(1 - t), n - t\bar{y}).$$

Let  $W \in Y \times_{Y/L} (N \setminus *) \times (N \setminus *)$  consist of the triples  $(y, n, m)$  such that

$$(m, \bar{y}) \in \Gamma_* \quad \text{and} \quad (1 - t)n - tm \in N \setminus * \text{ for all } t \in I.$$

To see the other required diagram for the dual pair commutes it is enough to show that the map

$$W \rightarrow V \times (Y \times_{Y/L} (N \setminus *))$$

defined by  $(y, n, m) \mapsto (n - m, g_{\tilde{m}, y} \tilde{m}, m)$ , where  $\tilde{m}$  is any lift of  $r(m)$  to  $Y$ , is homotopic to the map  $(y, n, m) \mapsto (m, y, n)$ . The homotopy

$$K(n, y, m, t) = ((1 - t)n + (2t - 1)m, \tilde{\omega}_{(\bar{y}, r(m))}(1 - t), (1 - t)m + tn)$$

demonstrates this.  $\square$

*Proof of Proposition 5.2.* Lemma 5.4 shows that for each  $x(H) \in \text{ob} \Pi_0(G, X)$

$$\overline{G|X}(x(H)): \Pi_0(G, X)^{\text{op}}(x(H), x(H)) \rightarrow \mathbf{Top}_*$$

is dualizable.

Define a functor

$$D(\overline{G|X}): \Pi_0(G, X) \rightarrow \mathbf{Top}_*$$

on objects by  $D(\overline{G|X})(x(H))$  is the dual of  $\overline{G|X}(x(H))$  constructed in Lemma 5.4. If  $R_a: G/H \rightarrow G/K$  is an isomorphism,  $D(\overline{G|X})(R_a)$  is induced by the action of  $a$  on  $X^H$ . If  $R_a$  is not an isomorphism the action is trivial.

The evaluation map for the dual pair  $(\overline{G|X}, D(\overline{G|X}))$  is a natural transformation

$$D(\overline{G|X}) \odot \overline{G|X} \rightarrow S^n \wedge \Pi_0(G, X)_+.$$

This is a compatible collection of maps

$$D(\overline{G|X})(x(H)) \wedge (X^K(y)/X^{>K}(y)) \rightarrow S^n \wedge \Pi_0(G, X)(x(H), y(K))_+.$$

If there are no equivariant maps  $G/H \rightarrow G/K$  then  $\Pi_0(G, X)(x(H), y(K))$  is empty. We only need to consider the cases where  $H$  is subconjugate to  $K$ . If  $H$  is conjugate to  $K$  the dual pair above defines the evaluation map. If  $H$  is subconjugate to  $K$  but not conjugate to  $K$ , naturality forces the evaluation map to be the zero map.

There is an inclusion of

$$\vee_{(H)} B(\overline{G|X})(x(H)), \Pi_0(G, X)(x(H), x(H))_+, D(\overline{G|X})(x(H))$$

into

$$B(\overline{G|X}, \Pi_0(G, X)_+, D(\overline{G|X})).$$

The coevaluation is the composite of the fold map

$$S^n \rightarrow \vee S^n,$$

the coevaluation maps for the spaces  $X^H(x)/X^{>H}(x)$ , and the inclusion above.

Since the coevaluation and evaluation maps are defined using the coevaluation and evaluation maps for dual pairs of Lemma 5.4, and because naturality forces many of the maps in the evaluation natural transformation to be zero, the required diagrams commute if they commute for each piece separately.  $\square$

*Remark 5.5.* In this section we do not need to assume the group  $G$  is finite. If  $G$  is an infinite discrete group Proposition 5.2 holds as long as

$$X^H/WH$$

is a compact ENR for all subgroups  $H$  of  $G$ .

## 6. THE GEOMETRIC LEFSCHETZ NUMBER

In this section we define the geometric Lefschetz number. This invariant serves as a transition between the local Lefschetz number defined in Section 1 and the global Lefschetz number defined in Section 7.

An equivariant map  $f: X \rightarrow X$  defines a compatible collection of  $WH$ -equivariant maps

$$f^H: X^H(x) \rightarrow X^H(f(x))$$

and

$$f^{>H}: X^{>H}(x) \rightarrow X^{>H}(f(x)).$$

The compatibility of these maps can be described using a natural transformation of the equivariant component space.

Let  $\Pi_0^f(G, X)$  be the  $\Pi_0(G, X)$ - $\Pi_0(G, X)$ -bimodule defined by

$$\Pi_0^f(G, X)(x(H), y(K)) = \Pi_0(G, X)(f(y(K)), x(H)).$$

Then  $\overline{G|X} \odot \Pi_0^f(G, X)$  is equivalent to  $\overline{G|X}(f(-))$ . An endomorphism  $f: X \rightarrow X$  induces a natural transformation

$$\overline{f}: \overline{G|X} \rightarrow \overline{G|X} \odot \Pi_0^f(G, X).$$

Since  $\overline{G|X}$  is dualizable the trace of  $\overline{f}$  is defined.

**Definition 6.1.** The *(extended) geometric Lefschetz number* of  $f$  is the trace of  $\overline{f}$ .

The (extended) geometric Lefschetz number is an element of  $\pi_0^s \left( \langle \Pi_0^f(G, X) \rangle_+ \right)$ . It is an invariant of the equivariant homotopy class of the map. This invariant is defined whenever Proposition 5.2 holds.

For  $x(H) \in \text{ob}\Pi_0(G, X)$  let

$$WH_{x,f} := \{g \in WH \mid [f(x)] = [xg] \in \pi_0(X^H)\}$$

and  $\langle WH_{x,f} \rangle$  denote the set of conjugacy classes of elements in  $WH_{x,f}$ . Let  $B(X)$  be the isomorphism classes of objects on  $\Pi_0(G, X)$ .

**Proposition 6.2.** *There is an isomorphism*

$$(6.3) \quad \delta: \pi_0^s \left( \langle \Pi_0^f(G, X) \rangle_+ \right) \rightarrow \mathbb{Z} \left( \coprod_{x(H) \in B(X)} \langle WH_{x,f} \rangle \right).$$

*The image of the (extended) geometric Lefschetz number of  $f$  under  $\delta$  is*

$$\sum_{y(K) \in B(X)} \sum_{g \in \langle WK_{y,f} \rangle} \left( \frac{1}{|C_{WH}(g)|} i(f|_{X_K(y)} g) \right) [y(K), g].$$

Recall that  $i$  is the nonequivariant index and  $C_{WH}(g) := \{h \in WH \mid hg = gh\}$ .

This proposition also identifies this invariant with the the local equivariant Lefschetz class of [21].

*Remark 6.4.* Using this proposition we see that the the (extended) geometric Lefschetz number detects more than fixed points. In later sections we will see that the (extended) global Lefschetz number, (extended) geometric Reidemeister trace and the (extended) global Reidemeister trace also have this property. In Sections 8 and 13 we will explain how to use these invariants to define invariants that detect only fixed points.

We divide the proof of this proposition into several lemmas.

For each object  $x(H)$  of  $\Pi_0(G, X)$ ,

$$\Pi_0^f(G, X)(x(H), x(H))$$

is an  $\Pi_0(G, X)(x(H), x(H))$ -  $\Pi_0(G, X)(x(H), x(H))$ -bimodule.

**Lemma 6.5.** *There is an isomorphism*

$$\langle \Pi_0^f(G, X) \rangle \rightarrow \coprod_{x(H) \in B(X)} \langle \Pi_0^f(G, X)(x(H), x(H)) \rangle.$$

*The image of the (extended) geometric Lefschetz number in*

$$\pi_0^s \left( \langle \Pi_0^f(G, X)(x(H), x(H)) \rangle_+ \right)$$

*is the trace of the natural transformation*

$$\overline{G|X}(x(H)) \rightarrow \overline{G|X}(x(H)) \odot \Pi_0^f(G, X)(x(H), x(H))$$

*induced by  $f$ .*

*Proof.* The functor  $\Pi(G, X) \rightarrow \Pi(G, X)$  induced by  $f$  satisfies the hypothesis of Lemma 3.9.

The decomposition of the trace corresponds to the decomposition of  $\langle \Pi_0^f(G, X) \rangle$ .  $\square$

**Lemma 6.6.** *There is an isomorphism*

$$\langle\langle \Pi_0^f(G, X)(x(H), x(H)) \rangle\rangle \rightarrow \langle\langle WH_{x,f} \rangle\rangle$$

for each  $x(H) \in B(X)$ .

*Proof.* The set  $\Pi_0(G, X)(f(x(H)), x(H))$  consists of the elements  $g$  of  $WH$  such that  $f(x)$  is in the same component of  $X^H$  as  $xg$ . The set  $\langle\langle \Pi_0^f(G, X) \rangle\rangle$  is defined to be the quotient of

$$\coprod \Pi_0(G, X)(f(x(H)), x(H)) = \coprod WH_{x,f}$$

by the relation  $g_1 \sim g_2$  if there are group elements  $h_1 \in \Pi_0(G, X)(x(H), y(K))$  and  $h_2 \in \Pi_0(G, X)(f(y(K)), x(H))$  such that  $h_1 h_2 = g_1$  and  $h_2 h_1 = g_2$ . This can also be written as  $g_1 \sim g_2$  if there is a group element  $h_2 \in \Pi_0(G, X)(f(y(K)), x(H))$  such that  $g_1 = h_2^{-1} g_2 h_2$ .  $\square$

The isomorphism  $\delta$  is the composite

$$\begin{aligned} \pi_0^s \left( \langle\langle \Pi_0^f(G, X) \rangle\rangle_+ \right) &\rightarrow \pi_0^s \left( \coprod_{x(H) \in B(X)} \langle\langle \Pi_0^f(G, X)(x(H), x(H)) \rangle\rangle \right) \\ &\rightarrow \pi_0^s \left( \coprod_{x(H) \in B(X)} \langle\langle WH_{x,f} \rangle\rangle \right) \\ &\cong \mathbb{Z} \left( \coprod_{x(H) \in B(X)} \langle\langle WH_{x,f} \rangle\rangle \right). \end{aligned}$$

Proposition 0.4 implies that for each conjugacy class of subgroups  $(H)$  and elements  $g$  and  $h$  that represent different conjugacy classes of elements in  $WH$  there are open disjoint neighborhoods  $U_g$  and  $U_h$  in  $X_{(H)}$  such that the fixed points of  $fg$  are in  $U_g$  and the fixed points of  $fh$  are in  $U_h$ .

**Lemma 6.7.** *The image of the geometric Lefschetz number under  $\delta$  is*

$$\sum_{x(H) \in B(X)} \sum_{g \in \langle\langle WH_{x,f} \rangle\rangle} i((f^H/WH)|_{U_g/WH})[x(H), g].$$

*Proof.* We use the notation from the proof of Lemma 5.4. Using Lemma 6.5 it is enough to show that this holds for a space  $X$  where  $G$  acts freely away from the base point. If  $x \in X$ ,  $\bar{x}$  is the image of  $x$  in  $X/G$ .

Embed  $X/G$  in a  $G$ -representation  $V$  with a neighborhood  $N$  and a retraction  $r: N \rightarrow X/G$ . Let

$$U = \{n \in N \mid (n, f/G(r(n))) \in \Gamma_*\}.$$

For each  $n \in U$ , there is a path from  $r(n)$  to  $f/G(r(n))$  that does not pass through the base point.

The trace of  $f$  is

$$\begin{array}{ccc} (V, V \setminus 0) & & \\ \cup \uparrow & & \\ (V, V \setminus B) & \longrightarrow & (V, (V \setminus X/G) \cup *) \\ & & \cup \uparrow \\ & & (U, U \setminus X/G) \xrightarrow{T} (V, V \setminus 0) \wedge \langle\langle G \rangle\rangle_+ \end{array}$$

where  $T(n) = (n - (f/G(r(n))), g_{x,f(x)})$  for  $x$  any point of  $X$  such that  $\bar{x} = r(n)$ .

The first component of this map is the nonequivariant index of  $f/G$ . The group element  $g_{x,f(x)}$  is as in Lemma 5.4. If  $f/G(y) = y$  for  $y \in X/G$ , there is an  $x \in X$  such that fixed by  $f g_{x,f(x)}(x) = x$ .  $\square$

Note that the points of  $X/G$  outside  $\text{II}U_g/G$  are mapped to the base point by the trace.

**Lemma 6.8.** *For  $f$  as above,*

$$i((f^H/WH)|_{U_g/WH}) = \frac{1}{|C_{WH}(g)|} i(f|_{X_H(y)g}).$$

*Proof.* To simplify the proof we assume the fixed points are isolated.

Let  $y$  be a fixed point of  $f/G$  that is contained in  $U_g$ . Suppose  $x \in X$  such that  $\bar{x} = y$ . Let  $H = G_x$ . Then there are exactly  $|C_{WH}(g)|$  points  $x$  of  $X^H$  such that  $f(x) = xg$  and  $\bar{x} = y$  in  $X/G$ . Since the fixed points of  $f$  are isolated and the action of  $WH$  on  $U_g$  is free, for each  $x$  there is a neighborhood  $V$  in  $X^H$ , homeomorphic to a neighborhood  $W$  in  $X^H/WH$ , such that

$$\begin{array}{ccc} V & \xrightarrow{f|_V} & V \\ \downarrow & & \downarrow \\ W & \xrightarrow{f/G|_W} & W \end{array}$$

commutes. The index of  $x$  with respect to  $f$  is the same as the index of  $y$  with respect to  $f/G$ .

Proposition 1.6 implies the index of  $f/WH|_{U_g/WH}$  is the sum of the indices of the fixed points. Then the index of  $f/WH|_{U_g/WH}$  is

$$\frac{i(f|_{U_g})}{|C_{WH}(g)|}.$$

$\square$

*Proof of Proposition 6.2.* The isomorphism  $\delta$  was defined above. Lemmas 6.7 and 6.8 complete the proof.  $\square$

**Examples.** We use Proposition 6.2 to compute the geometric Lefschetz number in the following examples. In all of these examples, and the examples in Section 10, the index of a fixed point  $x \in X$  with respect to an endomorphism  $f: X \rightarrow X$  is computed in  $X^{G_x}$ .

For an element  $g \in G$  and a subgroup  $H$  of  $G$  such that  $gHg^{-1} \subset H$ , we will use the notation  $g_{G/H}$  to indicate the  $G$ -map  $R_g: G/H \rightarrow G/H$ .

**Example 6.9.** There is an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $S^1 = \mathbb{R}/\mathbb{Z}$  by

$$t \cdot 1 = t + \frac{1}{2}.$$

The degree three map is equivariant with respect to this action. The fixed points of  $f$  and  $f \cdot 1$  are displayed below:

map	fixed points
$f$	$0, \frac{1}{2}$
$f \cdot 1$	$\frac{1}{4}, \frac{3}{4}$

Nonequivariantly, these fixed points all have index  $-1$ . The geometric Lefschetz number of this endomorphism is

$$-[0_{\mathbb{Z}/2\mathbb{Z}}] - [1_{\mathbb{Z}/2\mathbb{Z}}] \in \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}].$$

The degree 5 map is also equivariant with respect to this  $\mathbb{Z}/2\mathbb{Z}$  action. The fixed points of  $f$  and  $f \cdot 1$  are displayed below:

map	fixed points
$f$	$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$
$f \cdot 1$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$

Nonequivariantly, these fixed points all have index  $-1$ . The geometric Lefschetz number is

$$-2[0_{\mathbb{Z}/2\mathbb{Z}}] - 2[1_{\mathbb{Z}/2\mathbb{Z}}] \in \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}].$$

**Example 6.10.** Let  $S^2$  be the identification space  $(I \times I)/\sim$ , where  $(0, y) \sim (1, y)$  for all  $y \in I$  and  $(t_1, 0) \sim (t_2, 0)$  and  $(t_1, 1) \sim (t_2, 1)$  for all  $t_1, t_2 \in I$ . There is an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $S^2$  by

$$(x, y) \cdot 1 = (x, 1 - y).$$

Then  $(S^2)^{\mathbb{Z}/2\mathbb{Z}} = \{(t, \frac{1}{2})\} = S^1$ .

Define  $\psi_1: I \rightarrow I$  by

$$\psi_1(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \leq x < \frac{3}{4} \\ 2x - 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

This map is homotopic to the identity map. Define an endomorphism  $f: S^2 \rightarrow S^2$  by  $f(x, y) = (3x, \psi_1(y))$ . The map  $f$  is equivariant and taut. The fixed points of  $f$  and  $f \cdot 1$  are displayed below:

map	fixed points
$f$	$(t, 0), (t, 1), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$
$f \cdot 1$	$(0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$

Nonequivariantly, the fixed points  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, \frac{1}{2})$  each have index  $-1$ . The fixed points  $(t, 0)$  and  $(t, 1)$  have index 3. The geometric Lefschetz number of this map is

$$-2[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}] + 3[0_{\mathbb{Z}/2\mathbb{Z}}] + 0[1_{\mathbb{Z}/2\mathbb{Z}}] \in \mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})] \oplus \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}].$$

*Remark 6.11.* If we try to compute the geometric Lefschetz number using the map

$$g(x, y) = (3x, y)$$

rather than  $f$  we will get a different result. This is because the map  $g$  isn't taut. The same problem comes up in the following examples.

**Example 6.12.** Consider  $S^2$  with the same  $\mathbb{Z}/2\mathbb{Z}$  action as above. Define an endomorphism  $f$  by  $f(x, y) = (3x, 1 - \psi_1(y))$ . The fixed points of  $f$  and  $f \cdot 1$  are displayed below:

map	fixed points
$f$	$(0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$
$f \cdot 1$	$(t, 0), (t, 1), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$

The geometric Lefschetz number of this map is

$$-2[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}] + 0[0_{\mathbb{Z}/2\mathbb{Z}}] + 3[1_{\mathbb{Z}/2\mathbb{Z}}] \in \mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})] \oplus \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}].$$

**Example 6.13.** Define an action of  $\mathbb{Z}/3\mathbb{Z}$  on  $S^2$  by  $(x, y) \cdot 1 = (x + \frac{1}{3}, y)$ . Let  $\psi_2: I \rightarrow I$  be defined by

$$\psi_2(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{4} \\ 2x - \frac{1}{2} & \frac{1}{4} \leq x < \frac{3}{4} \\ 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

This map is homotopic to the identity map. Define a map  $g: S^2 \rightarrow S^2$  by

$$g(x, y) = (4x, \psi_2(x)).$$

This map is  $\mathbb{Z}/3\mathbb{Z}$  equivariant.

The fixed points of  $g$  and its translates under the action of  $\mathbb{Z}/3\mathbb{Z}$  are displayed below:

map	fixed points
$g$	$(t, 0), (t, 1), (0, \frac{1}{2}), (\frac{1}{3}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{2})$
$g \cdot 1$	$(t, 0), (t, 1), (\frac{2}{9}, \frac{1}{2}), (\frac{5}{9}, \frac{1}{2}), (\frac{8}{9}, \frac{1}{2})$
$g \cdot 2$	$(t, 0), (t, 1), (\frac{1}{9}, \frac{1}{2}), (\frac{4}{9}, \frac{1}{2}), (\frac{7}{9}, \frac{1}{2})$

Nonequivariantly, the fixed points  $(t, 0)$  and  $(t, 1)$  both have index 1. The remaining fixed points have index  $-1$ . Let  $\mathbb{Z}[(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 0]$  correspond to the point  $(t, 0)$  and  $\mathbb{Z}[(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 1]$  correspond to the point  $(t, 1)$ . The geometric Lefschetz number of  $g$  is

$$[0_{(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 0}] + [0_{(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 1}] - [0_{\mathbb{Z}/3\mathbb{Z}}] - [1_{\mathbb{Z}/3\mathbb{Z}}] - [2_{\mathbb{Z}/3\mathbb{Z}}]$$

in  $\mathbb{Z}[(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 0] \oplus \mathbb{Z}[(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 1] \oplus \mathbb{Z}[\mathbb{Z}/3\mathbb{Z}]$ .

**Example 6.14.** Let  $T^2$  be the space  $I \times I / \sim$  where  $(0, t) \sim (1, t)$  and  $(s, 0) \sim (s, 1)$ . There is a  $\mathbb{Z}/2\mathbb{Z}$  action on  $T^2$  defined by

$$(s, t) \cdot 1 = (s, 1 - t).$$

Then  $(T^2)^{\mathbb{Z}/2\mathbb{Z}} = \{(s, 0) | s \in I\} \amalg \{(s, \frac{1}{2}) | s \in I\}$ .

Define a map  $\psi_3: I \rightarrow I$  by

$$\psi_3(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{6} \\ 3x - \frac{1}{2} & \frac{1}{6} \leq x < \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} \leq x < \frac{2}{3} \\ 3x - \frac{3}{2} & \frac{2}{3} \leq x < \frac{5}{6} \\ 1 & \frac{5}{6} \leq x \leq 1 \end{cases}$$



This map is homotopic to the identity map. Define a map  $f: T^2 \rightarrow T^2$  by

$$f(x, y) = (3x, \psi_3(y)).$$

The fixed points of  $f$  and its translate under the action of  $\mathbb{Z}/2\mathbb{Z}$  are displayed below:

map	fixed points
$f$	$(0, 0), (0, \frac{1}{2}), (0, \frac{1}{4}), (0, \frac{3}{4}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{3}{4})$
$f \cdot 1$	$(0, 0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})$

Nonequivariantly, the fixed points  $(0, \frac{1}{4}), (0, \frac{3}{4}), (\frac{1}{2}, \frac{1}{4})$  and  $(\frac{1}{2}, \frac{3}{4})$  have index 1. The remaining fixed points have index  $-1$ .

Let  $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), 0]$  correspond to the circle  $\{(s, 0)\}$  and  $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), \frac{1}{2}]$  correspond to the circle  $\{(s, \frac{1}{2})\}$ . The geometric Lefschetz number of  $f$  is

$$-2[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), 0}] - 2[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), \frac{1}{2}}] + 2[0_{\mathbb{Z}/2\mathbb{Z}}] + 0[1_{\mathbb{Z}/2\mathbb{Z}}]$$

in  $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), 0] \oplus \mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), \frac{1}{2}] \oplus \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ .

**Example 6.15.** The symmetric group on three letters,  $\langle a, b | a^3, b^2, abab \rangle$ , acts on  $S^1$  by  $t \cdot a = t + \frac{1}{3}$ ,  $t \cdot b = 1 - t$ . The subgroups of the symmetric group on three letters are all cyclic and they are generated by

$$a, b, ba \text{ and } ba^2.$$

No points in  $S^1$  are fixed by  $a$  or  $a^2$ , the points  $0$  and  $\frac{1}{2}$  are fixed by  $b$ , the points  $\frac{1}{6}$  and  $\frac{2}{3}$  are fixed by  $ba$ , and the points  $\frac{1}{3}$  and  $\frac{5}{6}$  are fixed by  $ba^2$ .

Define a map  $f: S^1 \rightarrow S^1$  by

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < \frac{1}{18} \\ 12t - \frac{2}{3} & \text{if } \frac{1}{18} \leq t < \frac{1}{9} \\ \frac{2}{3} & \text{if } \frac{1}{9} \leq t < \frac{2}{9} \\ 12t & \text{if } \frac{2}{9} \leq t < \frac{5}{18} \\ \frac{1}{3} & \text{if } \frac{5}{18} \leq t < \frac{1}{3} \end{cases}$$

Extend to the rest of the unit interval using the action of  $a$ . This map is homotopic to the degree 4 endomorphism and is equivariant with respect to the action of  $b$ .

The translates of  $f$  by elements of  $S_3$  and their fixed points are displayed below:

map	fixed points
$f$	$0, \frac{2}{33}, \frac{3}{11}, \frac{1}{3}, \frac{13}{33}, \frac{20}{33}, \frac{2}{3}, \frac{8}{11}, \frac{31}{33}$
$f \cdot a$	$\frac{8}{33}, \frac{19}{33}, \frac{10}{11}$
$f \cdot a^2$	$\frac{1}{11}, \frac{14}{33}, \frac{25}{33}$
$f \cdot b$	$0, \frac{3}{13}, \frac{16}{39}, \frac{23}{39}, \frac{10}{13}$
$f \cdot ba$	$\frac{1}{13}, \frac{10}{39}, \frac{17}{39}, \frac{2}{3}, \frac{35}{39}$
$f \cdot ba^2$	$\frac{4}{39}, \frac{1}{3}, \frac{22}{39}, \frac{29}{39}, \frac{12}{13}$

With respect to the maps  $f$ ,  $f \cdot a$ , and  $f \cdot a^2$  the fixed points have index  $-1$ . With respect to the maps  $f \cdot b$ ,  $f \cdot ba$  and  $f \cdot ba^2$  the fixed points have index  $1$ .

Since no points of  $S^1$  are fixed by  $\langle a \rangle$  and the normalizer of  $\langle b \rangle$  is itself, the geometric Lefschetz number is an element of

$$\mathbb{Z}\langle\langle b \rangle / \langle b \rangle \rangle \oplus \mathbb{Z}\langle\langle ba \rangle / \langle ba \rangle \rangle \oplus \mathbb{Z}\langle\langle ba^2 \rangle / \langle ba^2 \rangle \rangle \oplus \mathbb{Z}\langle\langle S_3 \rangle \rangle.$$

Elements of the same order in  $S_3$  are conjugate, so  $\langle\langle S_3 \rangle \rangle$  consists of three elements.

The image of the geometric Lefschetz number in  $\mathbb{Z}\langle\langle b \rangle / \langle b \rangle \rangle$  is the sum of the indices of the fixed points in  $(S^1)^{\langle b \rangle}$ . The only fixed point is  $0$  and it has index  $1$ . The images of the geometric Lefschetz number in  $\mathbb{Z}\langle\langle ba \rangle / \langle ba \rangle \rangle$  and  $\mathbb{Z}\langle\langle ba^2 \rangle / \langle ba^2 \rangle \rangle$  are similar.

Let  $e$ ,  $b$ , and  $a$  represent the conjugacy classes of elements in  $S_3$ . The coefficient of  $e$  in  $\mathbb{Z}\langle\langle S_3 \rangle \rangle$  is the sum of the indices of fixed points of  $f$  that are not contained in  $(S^1)^{\langle b \rangle} \cup (S^1)^{\langle ba \rangle} \cup (S^1)^{\langle ba^2 \rangle}$  divided by the number of elements in  $S_3$  that commute with  $e$ . There are six fixed points, all of index  $-1$ , and all six elements in  $S_3$  commute with  $e$ .

The coefficients of  $a$  and  $b$  in  $\mathbb{Z}\langle\langle S_3 \rangle \rangle$  are similar. The elements  $e$ ,  $a$ , and  $a^2$  commute with  $a$ . The map  $f \cdot a$  has three fixed points, all of index  $-1$ . The only elements that commute with  $b$  are  $e$  and  $b$ . There are four fixed points of  $f \cdot b$ , all of index  $1$ .

The geometric Lefschetz number of  $f$  is

$$[b_{S_3/\langle b \rangle}] + [ba_{S_3/\langle ba \rangle}] + [ba^2_{S_3/\langle ba^2 \rangle}] - [e_{S_3}] - [a_{S_3}] + 2[b_{S_3}]$$

in  $\mathbb{Z}\langle\langle b \rangle / \langle b \rangle \rangle \oplus \mathbb{Z}\langle\langle ba \rangle / \langle ba \rangle \rangle \oplus \mathbb{Z}\langle\langle ba^2 \rangle / \langle ba^2 \rangle \rangle \oplus \mathbb{Z}\langle\langle S_3 \rangle \rangle$ .

## 7. THE GLOBAL LEFSCHETZ NUMBER

The classical Lefschetz fixed point theorem is a comparison of a local invariant and a global invariant. The equivariant Lefschetz fixed point theorem is also a comparison of local and global invariants. We have already defined the local invariant in Section 1. We gave a different description of the local invariant in Section 6. In this section we will define the global equivariant invariant. We will also complete the proof of an equivariant Lefschetz fixed point theorem by comparing the geometric and global invariants.

Let  $\mathbb{Z}\Pi_0(G, X)$  be the category with the same objects as  $\Pi_0(G, X)$ . The morphism group

$$\mathbb{Z}\Pi_0(G, X)(x(H), y(K))$$

is the free abelian group on the morphisms  $\Pi_0(G, X)(x(H), y(K))$ . Composing the equivariant component space functor with the cellular chain complex functor defines a distributor

$$C_*(\overline{G|X}): \mathbb{Z}\Pi_0(G, X)^{\text{op}} \rightarrow \text{Ch}_{\mathbb{Z}}.$$

**Lemma 7.1.** *If  $X$  is a finite  $G$ -CW complex the distributor  $C_*(\overline{G|X})$  is dualizable.*

*Proof.* We will describe two proofs of this result. One uses functoriality and the other is more direct.

Proposition 5.2 implies  $\overline{G|X}$  is dualizable. Since  $C_*(-)$  is strong symmetric monoidal, Proposition 2.6 implies  $C_*(\overline{G|X})$  is dualizable.

For an object  $x(H)$  of  $\Pi_0(G, X)$  let  $WH_x := \{h \in WH \mid [x] = [xh] \in \pi_0(X^H)\}$ . Then

$$C_*(\overline{G|X})(x(H))$$

is a finitely generated free  $\mathbb{Z}WH_x$ -module. This implies that  $C_*(\overline{G|X})(x(H))$  is dualizable with dual

$$\mathrm{Hom}_{\mathbb{Z}WH_x} \left( C_*(\overline{G|X})(x(H)), \mathbb{Z}WH_x \right).$$

This result and Corollary 3.8 imply that  $C_*(\overline{G|X})$  is dualizable.  $\square$

*Remark 7.2.* In this section we focus on the case of a finite  $G$ -CW complex. In the previous section we considered smooth closed  $G$ -manifolds and compact  $G$ -ENR's. These choices simplify notation, and in the rest of this paper we will continue to make similar choices to simplify notation. Except for Section 14, corresponding results hold for the other choices.

As we noted in Section 6, an equivariant map  $f: X \rightarrow X$  induces a natural transformation

$$\overline{f}_*: C_*(\overline{G|X}) \rightarrow C_*(\overline{G|X}) \odot \mathbb{Z}\Pi_0^f(G, X).$$

Where  $\mathbb{Z}\Pi_0^f(G, X)$  is a  $\mathbb{Z}\Pi_0(G, X)$ - $\mathbb{Z}\Pi_0(G, X)$ -bimodule defined by

$$\mathbb{Z}\Pi_0^f(G, X)(x(H), y(K)) = \mathbb{Z}\Pi_0(G, X)(f(y(K)), x(H)).$$

Since  $C_*(\overline{G|X})$  is dualizable the trace of  $\overline{f}_*$  is defined.

**Definition 7.3.** The *(extended) global Lefschetz number* of  $f$  is the trace of  $\overline{f}_*$ .

**Theorem 7.4** (Lefschetz fixed point theorem). *If  $f: X \rightarrow X$  is an equivariant map the (extended) global Lefschetz number of  $f$  is equal to the (extended) geometric Lefschetz number of  $f$ .*

*Proof.* The cellular chain complex functor

$$C_*: \mathbf{Top}_* \rightarrow \mathbf{Ch}_{\mathbb{Z}}$$

induces a functor from distributors enriched in based spaces to distributors enriched in chain complexes. Additionally, if  $X$  and  $Y$  are distributors enriched in topological spaces

$$\tau: C_*(X) \odot C_*(Y) \rightarrow C_*(X \odot Y)$$

is an isomorphism. The functor  $C_*$  also induces a map

$$\psi: \langle\langle C_*(P) \rangle\rangle \rightarrow C_*\langle\langle P \rangle\rangle$$

for any bimodule  $P$ . This shows  $C_*$  is a shadow functor.

The identification follows from Proposition 2.6.  $\square$

The following corollary is a consequence of Proposition 6.2 and Theorem 7.4.

**Corollary 7.5.** *If  $f: X \rightarrow X$  is an equivariant map and the set*

$$\{x \in X \mid \text{there is } g \in G \text{ such that } f(x) = xg\}$$

*is empty then the (extended) global Lefschetz number of  $f$  is zero.*

As in Section 6, we can also give another description of the global Lefschetz number. Recall the isomorphism

$$\delta: \mathbb{Z} \langle\langle \Pi_0(G, X) \rangle\rangle \rightarrow \mathbb{Z} (\Pi_{B(X)} \langle\langle WH_{x,f} \rangle\rangle)$$

of Proposition 6.2.

**Proposition 7.6.** *The image of the (extended) global Lefschetz number under  $\delta$  is*

$$\sum_{x(H) \in B(X)} \sum_{g \in \langle\langle WH_{x,f} \rangle\rangle} \frac{1}{|C_{WH}(g)|} \text{tr} (C_* (f^H g) : C_i (X_x^H, X_x^{>H}) \rightarrow C_i (X_x^H, X_x^{>H})) [x(H), g].$$

The trace  $\text{tr} (C_* (f^H g))$  is the usual trace of an endomorphism of a chain complex of abelian groups.

We divide the proof of this proposition into several lemmas.

**Lemma 7.7.** *The image of the (extended) global Lefschetz number in*

$$\langle\langle \Pi_0^f(G, X)(x(H), x(H)) \rangle\rangle$$

*is the trace of*

$$\bar{f}_* : C_* (\overline{G|X}) (x(H)) \rightarrow C_* (\overline{G|X}) (x(H)) \odot \mathbb{Z} \Pi_0^f(G, X)(x(H), x(H))$$

*as a module over  $\mathbb{Z} \Pi_0(G, X)(x(H), x(H))$ .*

*Proof.* This is analogous to Lemma 6.5. □

**Lemma 7.8.** *The trace of*

$$\bar{f}_* : C_* (\overline{G|X}) (x(H)) \rightarrow C_* (\overline{G|X}) (x(H)) \odot \mathbb{Z} \Pi_0^f(G, X)(x(H), x(H))$$

*as a module over  $\mathbb{Z} \Pi_0(G, X)(x(H), x(H))$  is*

$$\sum_i (-1)^{i \text{tr}_{WH_x}} (C_i (f^H) : C_i (X_x^H, X_x^{>H}) \rightarrow C_i (X_x^H, X_x^{>H})).$$

The trace  $\text{tr}_{WH}$  is the Hattori-Stallings trace. This lemma shows the (extended) global Lefschetz number coincides with the equivariant Lefschetz class defined in [17].

*Proof.* Since  $X$  is a finite  $G$ -CW complex,  $C_* (X_x^H, X_x^{>H})$  is a finite free  $\mathbb{Z} WH_x$ -chain complex for each  $x(H)$  in  $\Pi_0(G, X)$ .

Let  $\{e_{i,H}\}$  be a set of generators for the  $\mathbb{Z} WH_x$ -chain complex  $C_* (X_x^H, X_x^{>H})$ . Let  $\{e'_{i,H}\}$  denote the corresponding generators of the dual

$$\text{Hom}_{WH_x} (C_* (X_x^H, X_x^{>H}), \mathbb{Z} WH_x).$$

The coevaluation for the dual pair

$$(C_* (X_x^H, X_x^{>H}), \text{Hom}_{WH_x} (C_* (X_x^H, X_x^{>H}), \mathbb{Z} WH_x))$$

is defined by linearly extending the map

$$1 \mapsto \sum_{x(H) \in B(X)} \sum_i e_{i,H} \otimes e'_{i,H}.$$

The evaluation map is the usual evaluation.

The endomorphism  $C_*(\bar{f})$  can be described using a matrix  $(b_{i,j,H})$ ,  $b_{i,j,H} \in \mathbb{Z}\Pi_0(G, X)(x(H), x(H))$ . Let

$$\rho: \Pi_0(G, X)(x(H), x(H)) \rightarrow \langle\langle \Pi_0^f(G, X)(x(H), x(H)) \rangle\rangle \cong \langle\langle WH_{x,f} \rangle\rangle$$

be the quotient map. The image of 1 under the trace of  $C_*(\bar{f})$  is

$$\sum_i (-1)^{|e_{i,H}|} \rho(b_{i,i,H}).$$

This is the Hattori-Stallings trace.  $\square$

**Lemma 7.9.** *For  $f$  as above, the image of*

$$\sum_i (-1)^{i \operatorname{tr}_{WH_x}} (C_i(f^H) : C_i(X_x^H, X_x^{>H}) \rightarrow C_i(X_x^H, X_x^{>H}))$$

in  $\mathbb{Z}[g] \in \mathbb{Z}\langle\langle WH_{x,f} \rangle\rangle$  is

$$\sum_i (-1)^i \frac{1}{|C_{WH}(g)|} \operatorname{tr} (C_i(f^H g) : C_i(X_x^H, X_x^{>H}) \rightarrow C_i(X_x^H, X_x^{>H})).$$

*Proof.* We use the notation from the proof of the previous lemma.

If  $\{e_{i,H}\}$  is a set of generators for the  $\mathbb{Z}WH_x$ -chain complex  $C_*(X_x^H, X_x^{>H})$  then  $\{e_{i,H}g\}_{g \in WH_x}$  is a set of generators for  $C_*(X_x^H, X_x^{>H})$  as a chain complex of abelian groups. Using these generators the endomorphism  $C_*(\bar{f})$  can be described using a matrix  $(c_{g,i,h,j,H})$  of integers such that

$$f(e_{i,H}g) = \sum_{h \in WH} \sum_j c_{g,i,h,j,H} (e_j h).$$

Since the map is equivariant

$$c_{g,i,h,j,H} = c_{e,i,hg^{-1},j,H}.$$

The matrices  $(b_{i,j,H})$  and  $(c_{g,i,h,j,H})$  describe the same homomorphism so

$$b_{i,j,H} = \sum_{h \in WH} c_{e,i,h,j,H} h.$$

The image of 1 under the trace of  $C_*(\bar{f})$  is

$$\sum_i (-1)^{|e_{i,H}|} \rho(b_{i,i,H}).$$

This is the same as

$$\sum_i (-1)^{|e_{i,H}|} \sum_{h \in WH} c_{e,i,h,i} \rho(h) = \sum_{h \in WH} \left( \sum_i (-1)^{|e_{i,H}|} c_{e,i,h,i} \right) \rho(h).$$

The coefficient of  $g \in \langle\langle WH_{x,f} \rangle\rangle$  is

$$\sum_{\{h \in WH \mid \rho(h) = \rho(g)\}} \sum_i (-1)^{|e_{i,H}|} c_{e,i,h,i}.$$

The trace of  $C_*(\bar{f}g)$  as a map of  $\mathbb{Z}$ -modules is

$$\begin{aligned} \sum_i \sum_{h \in WH} (-1)^{|e_{i,H}|} c_{hg,i,h,i,H} &= \sum_i \sum_{h \in WH} (-1)^{|e_{i,H}|} c_{e,i,hg^{-1}h^{-1},i,H} \\ &= |C_{WH}(g)| \sum_i (-1)^{|e_{i,H}|} \sum_{\{k \in WH \mid \rho(k) = \rho(g)\}} c_{e,i,k,i,H}. \end{aligned}$$

□

**Examples.** We use Lemma 7.8 to compute the global Lefschetz numbers for the examples in Section 6.

**Example 7.10.** The component category of the space in Example 6.9 is equivalent to a category with one object and with endomorphisms given by  $\mathbb{Z}/2\mathbb{Z}$ . The component space functor is equivalent to a single chain complex with an action of  $\mathbb{Z}/2\mathbb{Z}$ . This chain complex is free over  $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$  with one generator in degree 0, called  $A$ , and one generator in degree 1, called  $B$ .

If  $f$  is the degree 3 endomorphism then

$$f_*(A) = A \quad f_*(B) = 2B + B(1_{\mathbb{Z}/2\mathbb{Z}}).$$

The trace of  $f_*$  is

$$-(2[0_{\mathbb{Z}/2\mathbb{Z}}] + [1_{\mathbb{Z}/2\mathbb{Z}}]) + [0_{\mathbb{Z}/2\mathbb{Z}}] = -[0_{\mathbb{Z}/2\mathbb{Z}}] - [1_{\mathbb{Z}/2\mathbb{Z}}]$$

in  $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ .

The degree 5 map is similar. In that case

$$f_*(A) = A \quad f_*(B) = 3B + 2B(1_{\mathbb{Z}/2\mathbb{Z}}).$$

The trace of  $f_*$  is

$$-(3[0_{\mathbb{Z}/2\mathbb{Z}}] + 2[1_{\mathbb{Z}/2\mathbb{Z}}]) + [0_{\mathbb{Z}/2\mathbb{Z}}] = -2[0_{\mathbb{Z}/2\mathbb{Z}}] - 2[1_{\mathbb{Z}/2\mathbb{Z}}]$$

in  $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ .

**Example 7.11.** The  $S^2$  in Example 6.10 can be decomposed into four 0-cells, six 1-cells, and four 2-cells. The 0-cells are the north and south pole and the points  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, \frac{1}{2})$ . The 1-cells divide  $S^2$  into northern and southern hemispheres and eastern and western hemispheres.

The component category is equivalent to a category with 2 objects. Call these objects  $a_1$  and  $a_2$ . The object  $a_1$  has a single endomorphism. There is also a single map from  $a_2$  to  $a_1$ . The endomorphisms of  $a_2$  are the group  $\mathbb{Z}/3\mathbb{Z}$ .

The chain complex of the component space corresponding to  $a_1$  is a free abelian group in each dimension. It has two generators in degree zero, call these generators  $A_1$  and  $A_2$ , and two generators in degree one, call these generators  $B_1$  and  $B_2$ . The generators  $A_1$  and  $A_2$  correspond to the points  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, \frac{1}{2})$ . The generators  $B_1$  and  $B_2$  are the 1-cells connecting these 2-cells.

The chain complex of the component space corresponding to  $a_2$  is a free module over  $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$  in each dimension. It has one generator in degree 0, call this generator  $A_3$ . It has two generators in degree one, call these generators  $B_3$  and  $B_4$ . It has two generators in degree two, call these generators  $C_1$  and  $C_2$ . We choose all of these generators in the northern hemisphere.

Then

$$\begin{aligned} f_*(A_1) &= A_1 & f_*(A_2) &= A_2 \\ f_*(B_1) &= 2B_1 + B_2 & f_*(B_2) &= B_1 + 2B_2. \end{aligned}$$

The trace of  $f_*$  on the component corresponding to  $a_1$  is

$$-(2 + 2) + (1 + 1) = -2.$$

For the generators corresponding to  $a_2$ ,

$$\begin{aligned} f_*(A_3) &= A_3 \\ f_*(B_3) &= B_3 & f_*(B_4) &= B_4 \\ f_*(C_1) &= 2C_1 + C_2 & f_*(C_2) &= C_1 + 2C_2. \end{aligned}$$

The trace of  $f_*$  on the component corresponding to  $a_2$  is

$$(2 + 2) - (1 + 1) + 1 = 3.$$

The global Lefschetz number is

$$-2[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}] + 3[0_{\mathbb{Z}/2\mathbb{Z}}] + 0[1_{\mathbb{Z}/2\mathbb{Z}}] \in \mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})] \oplus \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}].$$

**Example 7.12.** The global Lefschetz number of the endomorphism in Example 6.12 can be computed using the same decomposition as the example above.

The trace of  $f_*$  on the component corresponding to  $a_1$  is the same. On  $a_2$ ,

$$\begin{aligned} f_*(A_3) &= A_3(1_{\mathbb{Z}/2\mathbb{Z}}) \\ f_*(B_3) &= B_3(1_{\mathbb{Z}/2\mathbb{Z}}) & f_*(B_4) &= B_4(1_{\mathbb{Z}/2\mathbb{Z}}), \\ f_*(C_1) &= 2C_1(1_{\mathbb{Z}/2\mathbb{Z}}) + C_2(1_{\mathbb{Z}/2\mathbb{Z}}) & f_*(C_2) &= C_1(1_{\mathbb{Z}/2\mathbb{Z}}) + 2C_2(1_{\mathbb{Z}/2\mathbb{Z}}). \end{aligned}$$

The trace of  $f_*$  on the component corresponding to  $a_2$  is

$$(2[1_{\mathbb{Z}/2\mathbb{Z}}] + 2[1_{\mathbb{Z}/2\mathbb{Z}}]) - ([1_{\mathbb{Z}/2\mathbb{Z}}] + [1_{\mathbb{Z}/2\mathbb{Z}}]) + [1_{\mathbb{Z}/2\mathbb{Z}}] = 3[1_{\mathbb{Z}/2\mathbb{Z}}].$$

The global Lefschetz number is

$$-2[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}] + 0[0_{\mathbb{Z}/2\mathbb{Z}}] + 3[1_{\mathbb{Z}/2\mathbb{Z}}] \in \mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})] \oplus \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}].$$

**Example 7.13.** The component category for the space in Example 6.13 is equivalent to a category with 3 objects. Call these objects  $a_1$ ,  $a_2$  and  $a_3$ . Objects  $a_1$  and  $a_2$  each have a single endomorphism. There is also a single map from  $a_3$  to each of  $a_1$  and  $a_2$ . The endomorphisms of  $a_3$  are the group  $\mathbb{Z}/3\mathbb{Z}$ .

Decompose this  $S^2$  into two 0-cells, three 1-cells and three 2-cells. The 0-cells are the north and south poles. The 1-cells are evenly spaced longitude lines.

The chain complex of the component space corresponding to  $a_1$  is a free abelian group with a single generator in degree zero. Call this generator  $A_1$ . The chain complex of the component space corresponding to  $a_2$  is also a free abelian group on a single generator, call this generator  $A_2$ . The chain complex corresponding to  $a_3$  is a free module over  $\mathbb{Z}[\mathbb{Z}/3\mathbb{Z}]$  in each dimension. It has a single generator in degree one, call this generator  $B_1$ , and a single generator in degree two, call this generator  $C_1$ .

Then

$$\begin{aligned} f_*(A_1) &= A_1 & f_*(A_2) &= A_2 \\ f_*(B_1) &= B_1 \\ f_*(C_1) &= C_1(2(0_{\mathbb{Z}/3\mathbb{Z}}) + (1_{\mathbb{Z}/3\mathbb{Z}}) + (2_{\mathbb{Z}/3\mathbb{Z}})). \end{aligned}$$

The trace of  $f_*$  corresponding to  $a_1$  is 1. The trace of  $f_*$  corresponding to  $a_2$  is 1. The trace of  $f_*$  corresponding to  $a_3$  is

$$-(2[0_{\mathbb{Z}/3\mathbb{Z}}] + [1_{\mathbb{Z}/3\mathbb{Z}}] + [2_{\mathbb{Z}/3\mathbb{Z}}]) + [0_{\mathbb{Z}/3\mathbb{Z}}] = -[0_{\mathbb{Z}/3\mathbb{Z}}] - [1_{\mathbb{Z}/3\mathbb{Z}}] - [2_{\mathbb{Z}/3\mathbb{Z}}].$$

Let  $[(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 0]$  and  $[(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 1]$  correspond to the north and south poles respectively. The global Lefschetz number of  $f$  is

$$\begin{aligned} &[0_{(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 0}] + [0_{(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 1}] - [0_{\mathbb{Z}/3\mathbb{Z}}] - [1_{\mathbb{Z}/3\mathbb{Z}}] - [2_{\mathbb{Z}/3\mathbb{Z}}] \\ &\text{in } \mathbb{Z}[(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 0] \oplus \mathbb{Z}[(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}), 1] \oplus \mathbb{Z}[\mathbb{Z}/3\mathbb{Z}]. \end{aligned}$$

**Example 7.14.** The component category of  $T^2$  with the action of  $\mathbb{Z}/2\mathbb{Z}$  as in Example 6.14 is equivalent to a category with three objects,  $a_1$ ,  $a_2$ , and  $a_3$ . Objects  $a_1$  and  $a_2$  each have a single endomorphism. There is a single map from  $a_3$  to each of  $a_1$  and  $a_2$ . The endomorphisms of the object  $a_3$  are  $\mathbb{Z}/2\mathbb{Z}$ .

Decompose  $T^2$  using four 0-cells, eight 1-cells and four 2-cells. The 0-cells are the points  $(0, 0)$ ,  $(0, \frac{1}{2})$ ,  $(\frac{1}{2}, 0)$ , and  $(\frac{1}{2}, \frac{1}{2})$ . The one cells connect  $(0, 0)$  and  $(0, \frac{1}{2})$ ,  $(0, \frac{1}{2})$  and  $(0, 1)$ ,  $(\frac{1}{2}, 0)$  and  $(\frac{1}{2}, \frac{1}{2})$ ,  $(\frac{1}{2}, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$ ,  $(0, 0)$  and  $(\frac{1}{2}, 0)$ ,  $(\frac{1}{2}, 0)$  and  $(1, 0)$ ,  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, \frac{1}{2})$ , and  $(\frac{1}{2}, \frac{1}{2})$  and  $(1, \frac{1}{2})$ .

The chain complex associated to  $a_1$  is a free abelian group in each degree. It has two generators in degree 0, call these generators  $A_1$  and  $A_2$ , and two generators in degree 1, call these generators  $B_1$  and  $B_2$ . The chain complex associated to  $a_2$  is a free abelian group in each degree. It has two generators in degree 0, call these generators  $A_3$  and  $A_4$ , and two generators in degree 1, call these generators  $B_3$  and  $B_4$ . The chain complex associated to  $a_3$  is a free  $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$  module in each dimension. It has two generators in degree 1, call these generators  $B_5$  and  $B_6$ , and two generators in degree 2, call these generators  $C_1$  and  $C_2$ .

Then

$$\begin{aligned} f_*(A_i) &= A_i \text{ for } i = 1, 2, 3, 4, \\ f_*(B_1) &= 2B_1 + B_2 & f_*(B_2) &= B_1 + 2B_2 \\ f_*(B_3) &= 2B_3 + B_4 & f_*(B_4) &= B_3 + 2B_4 \\ f_*(B_5) &= B_5 & f_*(B_6) &= B_6 \\ f_*(C_1) &= 2C_1 + C_2 & f_*(C_2) &= C_1 + 2C_2. \end{aligned}$$

The trace of the component of  $f_*$  associated to  $a_1$  is

$$-(2 + 2) + (1 + 1) = -2.$$

The trace of the component of  $f_*$  associated to  $a_2$  is similar. The trace of the component of  $f_*$  associated to  $a_3$  is

$$(2 + 2) - (1 + 1) = 2.$$

The global Lefschetz number of  $f$  is

$$-2[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), 0}] + -2[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), \frac{1}{2}}] + 2[0_{\mathbb{Z}/2\mathbb{Z}}] + 0[1_{\mathbb{Z}/2\mathbb{Z}}]$$

in  $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), 0] \oplus \mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}), \frac{1}{2}] \oplus \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ .

**Example 7.15.** The component category for the space in Example 6.15 is equivalent to a category with seven objects. Denote these objects  $a_i$ ,  $i = 1, \dots, 7$ . The objects  $a_1$  and  $a_2$  correspond to the points of  $(S^1)^{\langle b \rangle}$ ,  $a_3$  and  $a_4$  correspond to the points of  $(S^1)^{\langle ba \rangle}$ , and  $a_5$  and  $a_6$  correspond to the points of  $(S^1)^{\langle ba^2 \rangle}$ . The object  $a_7$  is a point of  $S^1$ . Each object  $a_i$ ,  $i = 1, \dots, 6$  has a single endomorphism and there are no maps between any of these objects. The object  $a_7$  has endomorphisms  $S_3$  and there is a single map from  $a_7$  to each  $a_i$ ,  $i = 1, \dots, 6$ .

The chain complexes corresponding to each  $a_i$ ,  $i = 1, \dots, 6$ , are free abelian in each degree and have a single generator in degree 0. Let  $A_i$  be the generator of the chain complex corresponding to  $a_i$ . The chain complex corresponding to  $a_7$  is a free  $\mathbb{Z}S_3$  module in each degree. It has a single generator in degree 0, called  $A_7$ , and a single generator in degree 1, called  $B$ . The generator  $A_7$  corresponds to the point  $\frac{1}{6}$ . The generator  $B$  corresponds to the interval  $[0, \frac{1}{6}]$ .



Then

$$\begin{aligned} f_*(A_1) &= A_1 & f_*(A_2) &= A_1 \\ f_*(A_3) &= A_3 & f_*(A_4) &= A_3 \\ f_*(A_5) &= A_5 & f_*(A_6) &= A_5 \\ f_*(A_7) &= 0 \end{aligned}$$

$$f_*(B) = B - B(ba) + B(a) - B(ba^2).$$

The trace of the component of  $f$  corresponding to  $a_1$  is 1, that corresponding to  $a_2$  is 0. The traces for  $a_3$ ,  $a_4$ ,  $a_5$ , and  $a_6$  are similar. The trace of the component of  $f$  corresponding to  $a_7$  is

$$-[e - ba + a - ba^2] + 0 = -[e] - [a] + 2[b] \in \langle\langle S_3 \rangle\rangle.$$

The global Lefschetz number of  $f$  is

$$[b_{S_3/\langle b \rangle}] + [ba_{S_3/\langle ba \rangle}] + [ba_{S_3/\langle ba^2 \rangle}^2] - [e_{S_3}] - [a_{S_3}] + 2[b_{S_3}]$$

in  $\mathbb{Z}\langle\langle b \rangle/\langle b \rangle\rangle \oplus \mathbb{Z}\langle\langle ba \rangle/\langle ba \rangle\rangle \oplus \mathbb{Z}\langle\langle ba^2 \rangle/\langle ba^2 \rangle\rangle \oplus \mathbb{Z}\langle\langle S_3 \rangle\rangle$ .

## 8. COMPARISON OF LEFSCHETZ NUMBERS

In the previous two sections and in Section 1 we have defined three versions of the Lefschetz number. As noted before, the (extended) geometric and global Lefschetz numbers described more than fixed points. In this section we will describe how to use these invariants to define an invariant that only detects fixed points. We will also identify these new invariants with the local Lefschetz number and complete the proof of Theorem A from the introduction.

There is a map  $\pi_0^s \rightarrow \pi_0^{s,G}$  given by assigning a map

$$S^n \rightarrow S^n$$

the trivial  $G$  action. This map defines an action of  $\pi_0^s$  on  $\pi_0^{s,G}$ . Define a map

$$\xi: \oplus_{B(X)} \pi_0^s \rightarrow \pi_0^{s,G}$$

by  $\xi\left(\sum_{B(X)} c_{x(H)} x(H)\right) = \sum_{B(X)} c_{x(H)} \text{tr}(\text{id}_{G/H_+})$ .

The inclusion  $S^0 \rightarrow \langle\langle WH_{x,f} \rangle\rangle_+$  as the base point and the identity element defines a projection map

$$\oplus_{B(X)} \pi_0^s \left( \langle\langle WH_{x,f} \rangle\rangle_+ \right) \rightarrow \oplus_{B(X)} \pi_0^s$$

Let

$$\pi: \pi_0^s \left( \langle\langle \Pi_0^f(G, X) \rangle\rangle_+ \right) \rightarrow \pi_0^{s,G}$$

be the composite

$$\pi_0^s \left( \langle\langle \Pi_0^f(G, X) \rangle\rangle_+ \right) \xrightarrow{\delta} \oplus_{B(X)} \pi_0^s \left( \langle\langle WH_{x,f} \rangle\rangle_+ \right) \longrightarrow \oplus_{B(X)} \pi_0^s \xrightarrow{\xi} \pi_0^{s,G}.$$

**Definition 8.1.** The *geometric Lefschetz number* of  $f$  is the image of the (extended) geometric Lefschetz number of  $f$  under  $\pi$ .

**Proposition 8.2.** The *geometric Lefschetz number* of  $f$  is the *local Lefschetz number* of  $f$ .

The local Lefschetz number was defined in Section 1.

*Proof.* Proposition 6.2 implies the image of the geometric Lefschetz number in  $\oplus_{x(K) \in B(X)} \pi_0^s$  is

$$\sum_{B(X)} \left( \frac{i(f|_{X_K(y)})}{|WK|} \right) [y(K)].$$

Since the map is taut this is the same as

$$\sum_{B(X)} \left( \frac{i(f|_{X^{\kappa(y)}}) - i(f|_{X^{>\kappa(y)}})}{|WK|} \right) [y(K)].$$

The image of the geometric Lefschetz number in  $\pi_0^{s,G}$  is

$$\sum \left( \frac{i(f|_{X^{\kappa(y)}}) - i(f|_{X^{>\kappa(y)}})}{|WK|} \right) \text{tr}(G/K_+).$$

By [30, III.5.11], this is the local Lefschetz number.  $\square$

We can use a similar approach to define the global Lefschetz number.

Let  $I(G)$  be the set of conjugacy classes of subgroups of  $G$ . Define a map  $\mathbb{Z}(B(X)) \rightarrow \mathbb{Z}(I(G))$  by  $c_{x(H)}[x(H)] \mapsto c_{x(H)}(H)$ . Let

$$\pi: \mathbb{Z}\langle\langle \Pi_0^f(G, X) \rangle\rangle \rightarrow \mathbb{Z}(I(G))$$

be the composite

$$\mathbb{Z}\left(\langle\langle \Pi_0^f(G, X) \rangle\rangle\right) \xrightarrow{\delta} \mathbb{Z}(\Pi_{B(X)}\langle\langle WH_{x,f} \rangle\rangle) \longrightarrow \mathbb{Z}(B(X)) \longrightarrow \mathbb{Z}(I(G)).$$

Note that

$$\begin{array}{ccccc} \pi_0^s(\langle\langle \Pi_0^f(G, X) \rangle\rangle_+) & \xrightarrow{\pi} & \oplus_{B(X)} \pi_0^s & \longrightarrow & \oplus_{I(G)} \pi_0^s \longrightarrow \pi_0^{s,G} \\ \downarrow & & & & \uparrow \xi \\ \mathbb{Z}\langle\langle \Pi_0^f(G, X) \rangle\rangle & \xrightarrow{\pi} & \mathbb{Z}(I(G)) & & \end{array}$$

commutes.

**Definition 8.3.** The image of the (extended) global Lefschetz number of  $f$  under  $\pi$  is the *global Lefschetz number* of  $f$ .

Then Proposition 7.6 implies the following corollary.

**Corollary 8.4.** *The image of the global Lefschetz number of  $f$  under  $\pi$  is*

$$\sum_{I(G)} \frac{1}{|WH|} \text{tr}(C_*(f^H) : C_*(X^H, X^{>H}) \rightarrow C_*(X^H, X^{>H}))[H].$$

This description shows the equivariant Lefschetz number coincides with the equivariant Lefschetz class of [21] and the invariant  $L(f)$  in [33].

Theorem 7.4, Proposition 8.2 and the compatibility of the maps  $\pi$  imply the following theorem.

**Theorem A** (Equivariant Lefschetz fixed point theorem). [21] *If  $X$  is a compact  $G$ -ENR and  $f: X \rightarrow X$  is an equivariant map, the global Lefschetz number of  $f$  is equal to the local Lefschetz number of  $f$ .*

Since the local Lefschetz number of  $f$  is zero if  $f$  has no fixed points this statement implies the version of Theorem A from the introduction.

## 9. THE EQUIVARIANT UNIVERSAL COVER

In Section 5 we defined the equivariant component space and proved it was dualizable. In Sections 6 and 7 we used this distributor to define the geometric and global Lefschetz numbers. We also used this distributor to compare these invariants.

In this section we define a generalization of the equivariant component space and prove that it is dualizable. We will use this distributor to define the geometric and global Reidemeister traces and also to compare these invariants.

If  $x(H)$  is an object in the equivariant fundamental category of  $X$ , let  $\widetilde{X}_x^H$  be the universal cover of  $X_x^H$  thought of as homotopy classes of paths that start at  $x(eH)$ .

If  $\gamma: I \rightarrow X^K$  represents an element of  $\widetilde{X}_y^K$  and  $(R_a, w)$  is an object of

$$\Pi(G, X)(x(H), y(K)),$$

let  $(R_a, w) \circ \gamma$  be

$$(\gamma a)w(eH).$$

This is a path in  $X^H$  based at  $x(eH)$ . This composition defines an action of  $\Pi(G, X)(x(H), y(K))$  on  $\widetilde{X}(y(K))$ .

**Definition 9.1.** [6, I.10.13] The *equivariant universal cover* of a  $G$ -space  $X$  is the functor

$$\tilde{X}: \Pi(G, X)^{\text{op}} \rightarrow \mathbf{Top}_*$$

defined on objects by

$$\tilde{X}(x(H)) = \widetilde{X}_x^H.$$

The action of morphisms is defined above.

This functor is the starting point of our definition of the equivariant geometric Reidemeister trace. Like in Section 6, we will modify this functor before we define the Reidemeister trace.

Replace the functor  $\tilde{X}$  with the functor

$$\hat{X}: \Pi(G, X)^{\text{op}} \rightarrow \mathbf{Top}_*$$

defined by

$$\hat{X}(x(H)) = \widetilde{X}_x^H \cup C(\widetilde{X}_x^{>H})$$

where  $\widetilde{X}_x^{>H} = \{\gamma \in \widetilde{X}_x^H \mid \gamma(1) \in X^{>H}\}$ . The action of a morphism in  $\Pi(G, X)$  is that induced by the action on  $\tilde{X}$ .

**Proposition 9.2.** *If  $X$  is a compact  $G$ -ENR then  $\hat{X}$  is dualizable as a right  $\Pi(G, X)$ -module.*

We first prove a preliminary lemma.

**Lemma 9.3.** *For each object  $x(H)$  in  $\Pi(G, X)$   $\hat{X}(x(H))$  is dualizable as a module over  $\Pi(G, X)(x(H), x(H))$ .*

*Proof.* The group  $\Pi(G, X)(x(H), x(H))$  is a discrete group and  $\hat{X}(x(H))$  is cocompact. Then Lemma 5.4 implies  $\hat{X}(x(H))$  is Ranicki dualizable.  $\square$

*Proof of Proposition 9.2.* The dual spaces,  $D\hat{X}(x(H))$ , of  $\hat{X}(x(H))$  from Lemma 9.3 assemble to define a functor

$$D\hat{X} : \Pi(G, X) \rightarrow \mathbf{Top}_*.$$

The action of morphisms is induced by the action of each of the  $D\hat{X}(x(H))$ .

The evaluation for the dual pair  $(\hat{X}, D\hat{X})$  is a natural transformation

$$D\hat{X}(y(K)) \wedge \hat{X}(x(H)) \rightarrow S^n \wedge \Pi(G, X)(y(K), x(H))_+.$$

If  $K$  is not subconjugate to  $H$ ,  $\Pi(G, X)(x(H), y(K))$  is empty and there is nothing to define. If  $K$  is subconjugate to  $H$  but not conjugate to  $H$ , naturality implies this map is the constant map. It only remains to consider the case where  $H$  and  $K$  are conjugate. In that case we can use the evaluation defined above.

The coevaluation is a map

$$S^n \rightarrow B(\hat{X}, \Pi(G, X), D\hat{X}).$$

The space

$$\vee_{\text{ob}\Pi(G, X)} B(\hat{X}(x(H)), \Pi(G, X)(x(H), x(H)), D\hat{X}(x(H)))$$

is a subspace of  $B(\hat{X}, \Pi(G, X), D\hat{X})$ . The coevaluation is the composite of the pinch map  $S^n \rightarrow \vee S^n$ , the coevaluation maps constructed in Lemma 9.3 and this inclusion.

Since many of the components of the evaluation map are the constant map, checking that the required diagrams commute reduces to checking that each pair  $(\hat{X}(x(H)), D\hat{X}(x(H)))$  is a dual pair. This was checked in Lemma 9.3.  $\square$

*Remark 9.4.* For this proposition  $G$  does not have to be finite. It is enough that  $X^H/WH$  is a compact ENR for each subgroup  $H$  of  $G$  as in Lemma 5.3.

## 10. THE GEOMETRIC REIDEMEISTER TRACE

In this section we will define an equivariant generalization of the Nielsen number. The definition of this invariant is primarily motivated by the definition of the geometric Reidemeister trace in [24, 3.2] and the geometric Lefschetz number in Section 6. A different generalization of the Nielsen number is described in Section 15.

An equivariant map  $f : X \rightarrow X$  defines a natural transformation

$$\tilde{f} : \hat{X} \rightarrow \hat{X} \odot \Pi^f(G, X).$$

Where  $\Pi^f(G, X)$  is the  $\Pi(G, X)$ - $\Pi(G, X)$ -bimodule defined by

$$\Pi^f(G, X)(x(H), y(K)) = \Pi(G, X)(f(y(K)), x(H)).$$

Since  $\overline{G|X}$  is dualizable the trace of  $\tilde{f}$  is defined.

**Definition 10.1.** The (extended) geometric Reidemeister trace of  $f$ ,  $\overline{\mathcal{R}}^g(f)$ , is the trace of  $\tilde{f}$ .

The (extended) geometric Reidemeister trace is an element of

$$\pi_0^s(\langle\langle \Pi^f(G, X) \rangle\rangle_+).$$

It is an invariant of the equivariant homotopy class of  $f$ .

Let

$$\mathcal{O}(f) := \{y \in X \mid \text{there exists } k \in WG_y \text{ such that } f(y) = yk\}.$$

Define a map

$$\Theta: \mathcal{O}(f) \rightarrow \langle\langle \Pi^f(G, X) \rangle\rangle.$$

by  $\Theta(y) = (R_k: G/G_y \rightarrow G/G_y, c_{f(y)})$  where  $c_{f(y)}$  is the constant path at  $f(y)$ . Note that if  $f(y) = yk = yl$  then  $lk^{-1}$  fixes  $y$  and  $R_k = R_l$ .

**Proposition 10.2.** *If the fixed points of  $f$  are isolated, the image of the (extended) geometric Reidemeister trace under the isomorphism*

$$\pi_0^s(\langle\langle \Pi^f(G, X) \rangle\rangle_+) \cong \mathbb{Z}\pi_0(\langle\langle \Pi^f(G, X) \rangle\rangle)$$

is

$$\sum \frac{1}{|C_{WH}(g)|} i(f|_{X^H} g, \Theta^{-1}(R_g, \gamma)) [R_g, \gamma].$$

The map  $\Theta$  defines a relation on the fixed points of  $f$ . This generalizes the standard definition of fixed point classes. We will give other descriptions of this relation in Section 15.

We divide the proof of Proposition 10.2 into several lemmas.

**Lemma 10.3.** *There is an isomorphism*

$$\langle\langle \Pi^f(G, X) \rangle\rangle \rightarrow \Pi_{x(H) \in B(X)} \langle\langle \Pi^f(G, X)(x(H), x(H)) \rangle\rangle.$$

The image of the (extended) geometric Reidemeister trace in

$$\pi_0^s(\langle\langle \Pi^f(G, X)(x(H), x(H)) \rangle\rangle)$$

is the trace of

$$\tilde{f}(x(H)): \hat{X}(x(H), x(H)) \rightarrow \hat{X}(x(H), x(H)) \odot \Pi^f(G, X)(x(H), x(H)).$$

*Proof.* This is completely analogous to Lemma 6.5.  $\square$

**Lemma 10.4.** *Let  $X$  be a compact  $G$ -ENR and  $f: X \rightarrow X$  be an equivariant map. There is an open neighborhood  $U$  of the fixed points of  $f$  and a  $G$ -map*

$$\iota: U \rightarrow \Lambda_G^f X := \{\gamma \in X^I \mid f(\gamma(0)) = \gamma(1)\}$$

such that  $\iota(x)(0) = x$ ,  $\iota(x)(1) = f(x)$  and if  $f(x) = x$  then  $\iota(x)(t) = x$  for all  $t \in I$ .

*Proof.* This is a modification of the proof in [4, II.5.1].

Since  $X$  is a  $G$ -ENR there are equivariant maps

$$X \xrightarrow{i} W \xrightarrow{r} X$$

where  $W$  is an open invariant subset of a  $G$ -representation and  $r \circ i = \text{id}_X$ . Let  $V \subset X \times X$  be the points  $(x, y)$  where the line segment between  $i(x)$  and  $i(y)$  is contained in  $W$ . Note that the diagonal of  $X$  is in  $V$  and that  $(gx, gy) \in V$  for all  $g \in G$  if  $(x, y) \in V$ .

Define a  $G$ -map

$$H: V \times I \rightarrow X$$

by  $H(x, y, t) = r((1-t)i(x) + ti(y))$ . Then

$$U = \{x \in X \mid (x, f(x)) \in V\}$$

and  $\iota(x)(t) = H(x, f(x), t)$ .  $\square$

Using the open set  $U$  of this lemma and the open sets  $U_g$  of Proposition 0.4, for each  $(x, H)$  and group element  $g \in WH$ , there are open sets  $V_g \in X_H/WH$  such that the fixed points of  $f_H/WH$  are contained in the union of the sets  $V_g$  and there is a map

$$\alpha: \coprod V_g \rightarrow \langle\langle \Pi^f(G, X)(x(H), x(H)) \rangle\rangle$$

defined by  $\alpha(x) = (R_g, \iota(\tilde{x}))$  if  $x \in V_g$  and  $\tilde{x}$  is any lift of  $x$  to  $X$ .

If  $x \in \mathcal{O}(f)$ , then  $\Theta(x) = \alpha(\tilde{x})$ .

**Lemma 10.5.** *The image of the (extended) geometric Reidemeister trace in*

$$\pi_0^s \langle\langle \Pi^f(G, X)(x(H), x(H)) \rangle\rangle \cong \mathbb{Z} \langle\langle \Pi^f(G, X)(x(H), x(H)) \rangle\rangle$$

is

$$\sum_{\langle\langle \Pi^f(G, X) \rangle\rangle_{x(H), x(H)}} i(f|_{\alpha^{-1}(R_h, \sigma)}) [(R_h, \sigma)].$$

*Proof.* This is analogous to the proof of Proposition 6.2. We use the notation of Lemma 5.4. Using Lemma 10.3 we can restrict to the case of a free action except at a fixed base point.

Let

$$U = \{n \in N \setminus * \mid (n, f/G(r(n)) \in \Gamma_*\}.$$

If  $n \in U$  let  $\hat{n}$  be a lift of  $r(n)$  to  $X$ . There is an element  $g \in G$  and a path  $\hat{\omega}$  in  $X$  from  $f(\hat{n})$  to  $\hat{n}g$  such that  $\hat{\omega}$  projects to the path  $\omega$  defined in Lemma 5.4. Choose a lift  $\tilde{n}$  of  $\hat{n}$  to  $\tilde{X}$  and a path  $\tau$  from the base point  $*$  of  $X$  to  $f(*)$ . We think of  $\tilde{n}$  as a path in  $X$  from  $*$  to  $\hat{n}$ .

Note that  $(R_g, (\tilde{n}^{-1}g)\hat{\omega}f(\tilde{n})\tau)$  is an object of  $\Pi(G, X)$  such that

$$\tilde{n}(R_g, (\tilde{n}^{-1}g)\hat{\omega}f(\tilde{n})\tau) = (\tilde{n}g)(\tilde{n}^{-1}g)\hat{\omega}f(\tilde{n})\tau = \hat{\omega}f(\tilde{n})\tau.$$

This implies  $\gamma(t) = \hat{\omega}|_{[0, t]}f(\tilde{n})\tau$  is a path in  $\tilde{X}$  from  $\tilde{f}(\tilde{n})$  to  $\tilde{n}(R_g, (\tilde{n}^{-1}g)\hat{\omega}f(\tilde{n})\tau)$ .

The trace of  $\tilde{f}$  is the composite

$$\begin{array}{ccc} (V, V \setminus 0) & & \\ \cup \uparrow & & \\ (V, V \setminus B) & \longrightarrow & (V, (V \setminus X/L) \cup *) \\ & & \cup \uparrow \\ & & (U, U \setminus X/L) \xrightarrow{T} (V, V \setminus 0) \wedge \langle\langle \Pi^f(G, X)_+ \rangle\rangle \end{array}$$

where  $T(n) = (n - (f/G(r(n))), (R_g, (\tilde{n}^{-1}g)\hat{\omega}f(\tilde{n})\tau))$ . The first component is the nonequivariant index in  $X/G$ . If  $x \in \coprod V_g$  the second component agrees with the map  $\alpha$ . In this case  $\iota(\hat{n}) = \hat{\omega}$  and

$$(R_g, (\tilde{n}^{-1}g)\hat{\omega}f(\tilde{n})\tau)$$

is identified with  $(R_g, \hat{\omega})$  in  $\langle\langle \Pi^f(G, X) \rangle\rangle$  □

**Lemma 10.6.** *For  $f$  as above,*

$$i(f|_{\alpha^{-1}(R_h, \sigma)}) = \frac{1}{|C_{WH}(g)|} i(f|_{X^H} g, \Theta^{-1}(R_g, \gamma))$$

*Proof.* This is similar to the proof of Lemma 6.8. □

**Examples.** We use Proposition 10.2 to compute the (extended) geometric Reidemeister traces for the examples in Section 6. We use Proposition 11.5 to compute  $\langle\langle \Pi^f(G, X) \rangle\rangle$ .

**Example 10.7.** For the space and degree three endomorphism in Example 6.9

$$\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, S^1) \rangle\rangle$$

consists of two elements. If  $\gamma_0$  is the constant path at 0 and  $\gamma_1(t) = \frac{t}{2}$  then  $[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_0]$  and  $[1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1]$  represent these elements. Then

$$\begin{aligned}\Theta(0) &= \Theta\left(\frac{1}{2}\right) = [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_0], \\ \Theta\left(\frac{1}{4}\right) &= \Theta\left(\frac{3}{4}\right) = [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1].\end{aligned}$$

The indices were computed above, and so the extended geometric Reidemeister trace of this endomorphism is

$$-[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_0] - [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1] \in \mathbb{Z}\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, S^1) \rangle\rangle.$$

For the degree five endomorphism, the set

$$\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, S^1) \rangle\rangle$$

consists of four elements. If paths  $\gamma_i$  are defined by  $\frac{it}{2}$  for  $i = 0, 1, 2, 3$ , Then  $[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_0]$ ,  $[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_2]$ ,  $[1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1]$ , and  $[1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_3]$  represent these four elements. In this case,

$$\begin{aligned}\Theta(0) &= \Theta\left(\frac{1}{2}\right) = [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_0], \\ \Theta\left(\frac{1}{4}\right) &= \Theta\left(\frac{3}{4}\right) = [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_2], \\ \Theta\left(\frac{1}{8}\right) &= \Theta\left(\frac{5}{8}\right) = [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1], \\ \Theta\left(\frac{3}{8}\right) &= \Theta\left(\frac{7}{8}\right) = [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_3].\end{aligned}$$

The indices were computed above, and so the geometric Reidemeister trace of this endomorphism is

$$-[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_0] - [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1] - [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_2] - [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_3] \in \mathbb{Z}\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, S^1) \rangle\rangle.$$

**Example 10.8.** For the space and endomorphism in Example 6.10

$$\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, S^2) \rangle\rangle$$

consists of four elements. Let  $\gamma_0$  be the constant path in  $S^1$  based at 0,  $\gamma_1$  be the path in  $S^1$  defined by  $\gamma_1(t) = t$ ,  $\gamma_2$  be the constant path at a point of  $S^2$ , and  $\gamma_3$  be a path in  $S^2$  from a point  $x$  to  $x \cdot 1$ . Then  $[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0]$ ,  $[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1]$ ,  $[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_2]$  and  $[1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_3]$  represent the elements of  $\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, S^2) \rangle\rangle$ .

Then

$$\begin{aligned}\Theta\left(0, \frac{1}{2}\right) &= [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0], \\ \Theta\left(\frac{1}{2}, \frac{1}{2}\right) &= [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1] \\ \Theta(t, 0) &= \Theta(1, t) = [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_2].\end{aligned}$$

Since the index was computed above, the geometric Reidemeister trace of this endomorphism is

$$-[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1] + 3[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_2] \in \mathbb{Z}\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, S^2) \rangle\rangle.$$

**Example 10.9.** The geometric Reidemeister trace of the space in Example 6.12 is similar to the example above. In this case

$$\Theta(t, 0) = \Theta(1, t) = [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_3].$$

The geometric Reidemeister trace is

$$-[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1] + 3[1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_3] \in \mathbb{Z}\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, S^2) \rangle\rangle.$$

**Example 10.10.** For the space and endomorphism in Example 6.13 the geometric Lefschetz number and the geometric Reidemeister trace are the same.

**Example 10.11.** Define paths in  $T^2$  by

$$\begin{aligned} \gamma_{0,0}(t) &= (0, 0), \gamma_{0,1}(t) = (t, 0), \gamma_{\frac{1}{2},0}(t) = \left(0, \frac{1}{2}\right), \gamma_{\frac{1}{2},1}(t) = \left(t, \frac{1}{2}\right), \\ \alpha(t) &= (0, t), \alpha_{\frac{1}{2}}(t) = \left(0, \frac{t}{2} + \frac{1}{4}\right), \alpha_{\frac{3}{2}}(t) = \left(0, \frac{3t}{2} + \frac{1}{4}\right). \end{aligned}$$

For the space and endomorphism in Example 6.14, there are twelve elements in

$$\langle\langle \Pi^f(\mathbb{Z}/2\mathbb{Z}, T^2) \rangle\rangle.$$

They are represented by

$$\begin{array}{cccc} [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}] & [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}] & [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{\frac{1}{2},0}] & [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{\frac{1}{2},1}] \\ [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}, \gamma_{0,0}] & [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}, \alpha_1] & [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}, \gamma_{0,0}] & [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}, \alpha_1] \\ [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}, \alpha_{\frac{1}{2}}] & [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}, \alpha_{\frac{3}{2}}] & [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}, \alpha_{\frac{1}{2}}] & [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}, \alpha_{\frac{3}{2}}] \end{array}$$

Then

$$\Theta(0, 0) = [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}] \quad \Theta\left(\frac{1}{2}, 0\right) = [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}]$$

$$\Theta\left(0, \frac{1}{2}\right) = [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{\frac{1}{2},0}] \quad \Theta\left(\frac{1}{2}, \frac{1}{2}\right) = [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{\frac{1}{2},1}]$$

$$\Theta\left(0, \frac{1}{4}\right) = \Theta\left(0, \frac{3}{4}\right) = [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}, \gamma_{0,0}]$$

$$\Theta\left(\frac{1}{2}, \frac{1}{4}\right) = \Theta\left(\frac{1}{2}, \frac{3}{4}\right) = [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{\frac{1}{2},0}, \gamma_{0,0}].$$

The geometric Reidemeister trace of  $f$  is

$$\begin{aligned} &-[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{\frac{1}{2},0}] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{\frac{1}{2},1}] \\ &+ [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}, \gamma_{0,0}] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{\frac{1}{2},0}, \gamma_{0,0}] \end{aligned}$$

in  $\mathbb{Z}\langle\langle \Pi(\mathbb{Z}/2\mathbb{Z}, T^2) \rangle\rangle$ .

**Example 10.12.** Define paths in  $S^1$  by

$$\begin{aligned} \alpha_{\frac{1}{6}}(t) &= \frac{1}{6} & \alpha_{\frac{1}{3}}(t) &= \frac{1}{3} & \alpha_{\frac{1}{2}}(t) &= \frac{1}{2} \\ \alpha_{\frac{2}{3}}(t) &= \frac{2}{3} & \alpha_{\frac{5}{6}}(t) &= \frac{5}{6} \\ \beta(t) &= \frac{t}{3} \\ \gamma_i(t) &:= it \quad i = 0, 1, 2, 3, 4 \end{aligned}$$



For the space and endomorphism in Example 6.15 there are thirteen elements in  $\langle\langle \Pi^f(S_3, S^1) \rangle\rangle$ . They are represented by

$$\begin{array}{cccccc} [e_{S_3/\langle b \rangle}, \gamma_0] & [e_{S_3/\langle b \rangle}, \alpha_{\frac{1}{2}}] & [e_{S_3/\langle ab \rangle}, \alpha_{\frac{1}{6}}] & [e_{S_3/\langle ab \rangle}, \alpha_{\frac{2}{3}}] & & \\ [e_{S_3/\langle a^2b \rangle}, \alpha_{\frac{1}{3}}] & [e_{S_3/\langle a^2b \rangle}, \alpha_{\frac{5}{6}}] & & & & \\ & & & & & \\ [e_{S_3}, \gamma_0] & & & & & \\ [a_{S_3}, \beta] & & & & & \\ [b_{S_3}, \gamma_0] & [b_{S_3}, \gamma_1] & [b_{S_3}, \gamma_2] & [b_{S_3}, \gamma_3] & [b_{S_3}, \gamma_4] & \end{array}$$

Then

$$\begin{aligned} \Theta(0) &= [e_{S_3/\langle b \rangle}, \gamma_0] \\ \Theta\left(\frac{1}{3}\right) &= [e_{S_3/\langle a^2b \rangle}, \alpha_{\frac{1}{3}}] \\ \Theta\left(\frac{2}{3}\right) &= [e_{S_3/\langle ab \rangle}, \alpha_{\frac{2}{3}}] \\ \Theta\left(\frac{8}{33}\right) &= \Theta\left(\frac{19}{33}\right) = \Theta\left(\frac{10}{11}\right) = [a_{S_3}, \beta] \\ \Theta\left(\frac{3}{13}\right) &= \Theta\left(\frac{10}{13}\right) = [b_{S_3}, \gamma_1] \\ \Theta\left(\frac{16}{29}\right) &= \Theta\left(\frac{23}{29}\right) = [b_{S_3}, \gamma_3] \end{aligned}$$

The indices of  $0, \frac{1}{3}, \frac{2}{3}, \frac{8}{33}, \frac{19}{33}, \frac{10}{11}$  are  $-1$ . The indices of  $\frac{3}{13}, \frac{10}{13}, \frac{16}{29}$ , and  $\frac{23}{29}$  are  $1$ . Then the geometric Reidemeister trace of  $f$  is

$$[e_{S_3/\langle b \rangle}, \gamma_0] + [e_{S_3/\langle a^2b \rangle}, \alpha_{\frac{1}{3}}] + [e_{S_3/\langle ab \rangle}, \alpha_{\frac{2}{3}}] - [e_{S_3}, \gamma_0] - [a_{S_3}, \beta] + [b_{S_3}, \gamma_1] + [b_{S_3}, \gamma_3].$$

## 11. THE GLOBAL REIDEMEISTER TRACE

In this section we will define a generalization of the classical global Lefschetz number that corresponds to the geometric Reidemeister trace. This invariant is an equivariant generalization of the classical invariant defined in [11].

Let  $\mathbb{Z}\Pi(G, X)$  be the category with the same objects as  $\Pi(G, X)$  and let

$$\mathbb{Z}\Pi(G, X)(x(H), y(K))$$

be the free abelian group on  $\Pi(G, X)(x(H), y(K))$ . Define a right  $\mathbb{Z}\Pi(G, X)$  module  $C(\hat{X})$  by

$$C(\hat{X})(x(H)) = C_*(\widetilde{X_x^H}, \widetilde{X_x^{>H}}).$$

There is an induced action of  $\Pi(G, X)$  on  $C(\hat{X})$ .

**Lemma 11.1.** *If  $X$  is a finite  $G$ -CW complex,  $C(\hat{X})$  is dualizable as a distributor over  $\mathbb{Z}\Pi(G, X)$ .*

*Proof.* Like Lemma 7.1 this lemma follows from Proposition 9.2 and Proposition 2.6 or, more directly, from Corollary 3.8.  $\square$

An equivariant map

$$f: X \rightarrow X$$

induces a natural transformation

$$\hat{f}_*: C_*(\hat{X}) \rightarrow C_*(\hat{X}) \odot \mathbb{Z}\Pi^f(G, X).$$

Here  $\mathbb{Z}\Pi^f(G, X)$  is the  $\mathbb{Z}\Pi(G, X)$ - $\mathbb{Z}\Pi(G, X)$ -bimodule defined on objects by

$$\mathbb{Z}\Pi^f(G, X)(x(H), y(K)) = \mathbb{Z}\Pi(G, X)(f(y(K)), x(H)).$$

Since  $C(\hat{X})$  is dualizable the trace of  $\hat{f}$  is defined.

**Definition 11.2.** The (extended) global Reidemeister trace of  $f$ ,  $\overline{\mathcal{R}}^{gl}(f)$ , is the trace of  $\tilde{f}$ .

**Proposition 11.3.** The cellular chain complex functor defines an isomorphism

$$\pi_0^s(\langle\langle\Pi^f(G, X)\rangle\rangle_+) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}\langle\langle\Pi^f(G, X)\rangle\rangle)$$

and under this isomorphism  $\overline{\mathcal{R}}^g(f) = \overline{\mathcal{R}}^{gl}(f)$ .

*Proof.* The (extended) geometric Reidemeister trace is the trace of

$$\hat{X} \xrightarrow{\tilde{f}} \hat{X} \odot \Pi^f(G, X) .$$

The (extended) global Reidemeister trace is the trace of

$$C_*(\hat{X}) \xrightarrow{C_*(\tilde{f})} C_*(\hat{X} \odot \Pi^f(G, X)) \xleftarrow{\sim} C_*(\hat{X}) \odot \Pi^f(G, X) .$$

Proposition 2.6 implies  $\overline{\mathcal{R}}^g(f) = \overline{\mathcal{R}}^{gl}(f)$ .  $\square$

Like in the previous sections, we can give another description of this invariant.

If  $x(H)$  is an element of  $B(X)$ , let  $x$  denote  $x(eH)$ . For each  $x(H) \in B(X)$  choose a path  $\tau$  in  $X^H$  from  $x$  to  $f(x)$ . For  $k \in WH$  and  $x \in X^H$ , let  $\pi_1(X^H, x, xk)$  be the homotopy classes of paths, with end points fixed, from  $x$  to  $xk$ . Define an equivalence relation on  $\pi_1(X^H, x, xk)$  by  $\alpha \sim \beta$  if there is a group element  $h$  such that  $k = h^{-1}kh$  and a path  $\gamma$  from  $x$  to  $xh$  in  $X^H$  such that  $\beta$  is homotopic to

$$(11.4) \quad (\gamma^{-1}k)(\alpha h)(\tau^{-1}h)f(\gamma)\tau$$

in  $X^H$  with endpoints fixed.

Let  $\langle\langle\pi_1(X^H, k)\rangle\rangle$  denote  $\pi_1(X^H, x, xk)$  modulo this relation. Let  $\langle\langle\pi_1(X^H)\rangle\rangle$  denote the usual semiconjugacy classes of  $\pi_1(X^H)$  with respect to the map induced by  $f^H$ . For each  $k$  such that  $x$  and  $xk$  are in the same connected component of  $X^H$ , there is a map

$$q: \langle\langle\pi_1(X^H)\rangle\rangle \rightarrow \langle\langle\pi_1(X^H, k)\rangle\rangle$$

defined by composing with a fixed path from  $xk$  to  $x$ . In many cases this map is not an isomorphism. Let  $\langle\langle\Pi^f(G, X)\rangle\rangle$  denote

$$\coprod_{x(H) \in B(X)} \coprod_{k \in \langle\langle WH_{x,f} \rangle\rangle} \langle\langle\pi_1(X^H, k)\rangle\rangle.$$

The map  $q$  defines a map

$$q: \langle\langle\pi_1(X_x^H)\rangle\rangle \rightarrow \langle\langle\Pi^f(G, X)\rangle\rangle$$

for each  $x(H) \in B(X)$ .

**Proposition 11.5.** There is an isomorphism

$$\delta: \langle\langle\Pi^f(G, X)\rangle\rangle_+ \rightarrow \langle\langle\Pi^f(G, X)\rangle\rangle.$$

The image of the (extended) global Reidemeister trace in  $\langle\langle\pi_1(X_x^H, k)\rangle\rangle$  is

$$q \left( \frac{1}{|C_{WH}(k)|} \text{tr}_{\mathbb{Z}\pi_1(X^H)} \left( C_*(\widetilde{f^H}k): C_*\left(\widetilde{X_x^H}, \widetilde{X_x^{>H}}\right) \rightarrow C_*\left(\widetilde{X_x^H}, \widetilde{X_x^{>H}}\right) \right) \right).$$

The trace  $\text{tr}_{\mathbb{Z}\pi_1(X^H)}$  is the Hattori-Stallings trace. We divide the proof of this proposition into several lemmas.

Recall the isomorphism

$$\mathbb{Z}\langle\langle\Pi^f(G, X)\rangle\rangle \rightarrow \oplus_{x(H) \in B(X)} \mathbb{Z}\langle\langle\Pi^f(G, X)(x(H), x(H))\rangle\rangle$$

of Lemma 10.3.

**Lemma 11.6.** *The image of the (extended) global Reidemeister trace in*

$$\langle\langle\Pi_0^f(G, X)(x(H), x(H))\rangle\rangle$$

*is*

$$\sum (-1)^i \text{tr}_{\Pi(G, X)(x(H), x(H))} \left( C_i(\tilde{f}^H) : C_i(\widetilde{X_x^H}, \widetilde{X_x^{>H}}) \rightarrow C_i(\widetilde{X_x^H}, \widetilde{X_x^{>H}})^f \right).$$

In this lemma and in Lemma 11.8 we use the superscript  $f$  to indicate the action of the category  $\Pi(G, X)$  on the target of this morphism is twisted by  $f$  as before.

*Proof.* This is analogous to Lemma 7.7.  $\square$

Since the proofs of the following two lemmas are not difficult, but the notation is complicated, we only include an outline here.

**Lemma 11.7.** *There is an isomorphism*

$$\delta : \langle\langle\Pi^f(G, X)\rangle\rangle_+ \rightarrow \langle\langle\Pi^f(G, X)\rangle\rangle.$$

*Proof.* Define a map

$$\Upsilon : \coprod_{B(X)} \coprod_{k \in \langle\langle WH_{x,f} \rangle\rangle} \pi_1(X^H, x, xk) \rightarrow \coprod_{y(K) \in \text{ob}\Pi(G, X)} \Pi(G, X)(f(y(K)), y(K))$$

by  $\Upsilon(\alpha) = (R_k : G/H \rightarrow G/H, \alpha\tau^{-1})$  for  $\alpha \in \pi_1(X^H, x, xk)$ . The equivalence relations that define  $\langle\langle\Pi^f(G, X)\rangle\rangle$  and  $\langle\langle\Pi^f(G, X)\rangle\rangle_+$  are compatible, and so  $\Upsilon$  induces a map

$$\Upsilon : \langle\langle\Pi^f(G, X)\rangle\rangle_+ \rightarrow \langle\langle\Pi^f(G, X)\rangle\rangle.$$

If  $\phi$  is a path from  $f(y)$  to  $ym$ , define a map

$$\Psi : \coprod_{y(K) \in \text{ob}\Pi(G, X)} \Pi(G, X)(f(y(K)), y(K)) \rightarrow \coprod_{B(X)} \coprod_{k \in \langle\langle WH_{x,f} \rangle\rangle} \pi_1(X^H, x, xk)$$

by

$$\Psi(R_m : G/K \rightarrow G/K, \phi) := (\sigma^{-1}k)(\phi j)f(\sigma) \in \pi_1(X^H, x, xk)$$

where  $j \in WH$  satisfies  $j^{-1}mj = k$  and  $\sigma$  is any path in  $X^H$  from  $x \in B(X)$  to  $yj$ . It is not hard to show  $\Psi$  induces a well defined map

$$\Psi : \langle\langle\Pi^f(G, X)\rangle\rangle_+ \rightarrow \langle\langle\Pi^f(G, X)\rangle\rangle.$$

The induced maps  $\Upsilon$  and  $\Psi$  define an isomorphism between  $\langle\langle\Pi^f(G, X)\rangle\rangle$  and  $\langle\langle\Pi^f(G, X)\rangle\rangle_+$ .  $\square$

**Lemma 11.8.** *For  $f$  as above, the image of*

$$\text{tr}_{\Pi(G, X)(x(H), x(H))} \left( C_* (\tilde{f}^H) : C_* (\widetilde{X_x^H}, \widetilde{X_x^{>H}}) \rightarrow C_* (\widetilde{X_x^H}, \widetilde{X_x^{>H}})^f \right)$$

in  $\mathbb{Z}\langle\pi_1(X^H, g)\rangle$  is

$$q\left(\frac{1}{|C_{WH}(g)|}\mathrm{tr}_{\mathbb{Z}\pi_1(X^H)}\left(C_*(\widetilde{f^H}g): C_*\left(\widetilde{X_x^H}, \widetilde{X_x^{>H}}\right) \rightarrow C_*\left(\widetilde{X_x^H}, \widetilde{X_x^{>H}}\right)^f\right)\right).$$

*Proof.* The proof of this lemma is very similar to the proof of Lemma 7.9.  $\square$

**Examples.** We use the cellular decompositions described in Section 7 to compute the global Lefschetz numbers for the examples in Section 6. The cellular structure of the universal cover is induced from that in the base by choosing a lift of one of the zero cells.

**Example 11.9.** For the space and degree three endomorphism in Example 6.9 let  $\gamma_0$  be the constant loop at zero,  $\gamma$  be the loop  $\gamma(t) = t$ , and  $\gamma_1(t) = \frac{t}{2}$ . Then

$$\begin{aligned}\tilde{f}_*(A) &= A \\ \tilde{f}_*(B) &= B + B(1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1) + B(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma).\end{aligned}$$

The trace of  $\tilde{f}_*$  is

$$-(1 + [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma]) + 1 = -[1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1] - [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma].$$

In the shadow  $[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma]$  is identified with  $[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_0]$ .

For the degree five endomorphism

$$\begin{aligned}\tilde{f}_*(A) &= A \\ \tilde{f}_*(B) &= B + B(1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1) + B(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma) + B(1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1\gamma) + B(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma^2).\end{aligned}$$

The trace of  $\tilde{f}_*$  is

$$-(1 + [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma] + [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1\gamma] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma^2]) + 1$$

In the shadow this is identified with

$$-[1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1] - [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma] - [1_{\mathbb{Z}/2\mathbb{Z}}, \gamma_1\gamma] - [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma^2].$$

**Example 11.10.** For the space and endomorphism in Example 6.10 let  $\gamma_0$  be constant path at  $(0, \frac{1}{2})$  and let  $\gamma_1(t) = (t, \frac{1}{2})$ .

The fundamental category is equivalent to a category with two objects corresponding to the two objects of the component category. Call these objects  $a_1$  and  $a_2$ . On  $a_2$ , the global Reidemeister trace is the same as the global Lefschetz number. For  $a_1$ ,

$$\tilde{f}_*(A_1) = A_1(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0) \quad \tilde{f}_*(A_2) = A_2(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1)$$

$$\tilde{f}_*(B_1) = B_1(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0) + B_2(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0) + B_1(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1)$$

$$\tilde{f}_*(B_2) = B_2(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1) + B_1(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1^2) + B_2(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1^2).$$

The trace of the component of  $f$  corresponding to  $a_1$  is

$$\begin{aligned}& -([0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0] + [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1] + [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1] + [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1^2]) + \\ & \quad ([0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0] + [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1]) \\ & = -[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1^2].\end{aligned}$$

In the shadow  $[\gamma_1^2]$  is identified with  $\gamma_0$ .

If  $\gamma_2$  is the constant path at some point of  $S^2$ , the global Reidemeister trace of this endomorphism is

$$-[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_0] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_1] + 3[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_2]$$

**Example 11.11.** The fundamental category of the space in Example 6.12 is equivalent to a category with two objects,  $a_1$  and  $a_2$ . On  $a_1$ , the global Reidemeister trace is the same as the previous example. On  $a_2$ , the global Lefschetz number is the same as the global Reidemeister trace.

**Example 11.12.** The global Reidemeister trace for the space and endomorphism in Example 6.13 is the same as the global Lefschetz number.

**Example 11.13.** Let

$$\gamma_{0,0}(t) = (0, 0), \gamma_{0,1}(t) = (t, 0)$$

be paths in  $T^2$ . For the space and endomorphism in Example 6.14,

$$\begin{aligned} f_*(A_1) &= A_1(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}) & f_*(A_2) &= A_2(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}) \\ f_*(A_3) &= A_3(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}) & f_*(A_4) &= A_4(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}) \end{aligned}$$

$$\begin{aligned} f_*(B_1) &= B_1(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}) + B_2(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}) + B_1(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}) \\ f_*(B_2) &= B_2(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}) + B_1(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}^2) + B_2(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}^2) \\ f_*(B_3) &= B_3(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}) + B_4(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}) + B_3(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}) \\ f_*(B_4) &= B_4(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}) + B_3(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}^2) + B_4(0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}^2), \end{aligned}$$

$$f_*(B_5) = B_5(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}) \quad f_*(B_6) = B_6(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1})$$

$$\begin{aligned} f_*(C_1) &= C_1(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}) + C_2(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}) + C_1(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}) \\ f_*(C_2) &= C_2(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}) + C_1(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}^2) + C_2(0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}^2). \end{aligned}$$

The trace of  $f$  corresponding to  $a_1$  is

$$\begin{aligned} & -([0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}] + [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}] + [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}] + [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}^2]) \\ & + ([0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,0}] + [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}]) \\ & = -[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}^2]. \end{aligned}$$

The trace of  $f$  corresponding to  $a_2$  is

$$-[0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}] - [0_{(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})}, \gamma_{0,1}^2].$$

The trace of  $f$  corresponding to  $a_3$  is

$$([0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}^2]) - ([0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,0}] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}]).$$

This is  $[0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}] + [0_{\mathbb{Z}/2\mathbb{Z}}, \gamma_{0,1}^2]$ .

**Example 11.14.** The computation of the global Reidemeister trace for the endomorphism in Example 6.15 is the similar to the computation of the global Lefschetz number. Let

$$\begin{aligned} \alpha_{\frac{1}{6}}(t) &= \frac{1}{6} & \alpha_{\frac{1}{3}}(t) &= \frac{1}{3} & \alpha_{\frac{1}{2}}(t) &= \frac{1}{2} \\ \alpha_{\frac{2}{3}}(t) &= \frac{2}{3} & \alpha_{\frac{5}{6}}(t) &= \frac{5}{6} \\ \beta(t) &= \frac{t}{3} \\ \gamma_i(t) &:= it \quad i = 0, 1, 2, 3, 4 \end{aligned}$$

Then

$$\begin{aligned} f_*(A_1) &= A_1(e_{S_3/\langle b \rangle}, \gamma_0) & f_*(A_2) &= A_1(e_{S_3/\langle b \rangle}, \gamma_0) \\ f_*(A_3) &= A_3(e_{S_3/\langle ba \rangle}, \alpha_{\frac{1}{6}}) & f_*(A_4) &= A_3(e_{S_3/\langle ba \rangle}, \alpha_{\frac{1}{6}}) \\ f_*(A_5) &= A_5(e_{S_3/\langle ba^2 \rangle}, \alpha_{\frac{1}{3}}) & f_*(A_6) &= A_5(e_{S_3/\langle ba^2 \rangle}, \alpha_{\frac{1}{3}}) \\ f_*(A_7) &= 0 \end{aligned}$$

$$f_*(B) = B(e_{S_3}, \gamma_0) - B(ba_{S_3}, \alpha_{\frac{1}{6}}) + B(a_{S_3}, \beta) - B(ba_{S_3}^2, \alpha_{\frac{1}{3}}).$$

The trace of  $f_*$  is

$$([e_{S_3/\langle b \rangle}, \gamma_0] + [e_{S_3/\langle ba \rangle}, \alpha_{\frac{1}{6}}] + [e_{S_3/\langle ba^2 \rangle}, \alpha_{\frac{1}{3}}]) - ([e_{S_3}, \gamma_0] - [ba_{S_3}, \alpha_{\frac{1}{6}}] + [a_{S_3}, \beta] - [ba_{S_3}^2, \alpha_{\frac{1}{3}}])$$

In the shadow  $[ba_{S_3}, \alpha_{\frac{1}{6}}]$  is identified with  $[b_{S_3}, \gamma_1]$  and  $[ba_{S_3}^2, \alpha_{\frac{1}{3}}]$  is identified with  $[b_{S_3}, \gamma_3]$ .

## 12. THE LOCAL REIDEMEISTER TRACE

In this section we will define the refinement of the local Lefschetz number. Like the invariants in the previous two sections, the primary difference between the local Reidemeister trace and the local Lefschetz number is that the local Reidemeister trace includes information about the fundamental group. Unlike the other versions of the Reidemeister trace, the local Reidemeister trace is not defined using the isotropy subspaces. This invariant is defined, like the local Lefschetz number, using the equivariant stable homotopy category.

**Lemma 12.1.** *Suppose  $X$  is a closed smooth  $G$ -manifold that embeds in a  $G$ -representation  $V$  with normal bundle  $\nu$ . Then*

$$((\mathcal{P}(X), s)_+, T_X s^* S^\nu)$$

*is a dual pair.*

The space  $S^\nu$  is the fiberwise one point compactification of  $\nu$ . It is an ex-space over  $X$  with projection given by the map  $\nu \rightarrow X$ . The parametrized space  $T_X s^* S^\nu$  is defined to be

$$L(\mathcal{P}X, t \times s)_+ \odot S^\nu.$$

There is a similar statement for  $G$ -ENR's.

*Proof.* Note that  $(\mathcal{P}(X), s)_+ \cong S_X^0 \odot R(\mathcal{P}X, t \times s)_+$ .

Proposition 16.4 implies that  $(S_X^0, S^\nu)$  is a dual pair. Composing this dual pair with the dual pair

$$(R(\mathcal{P}(X), t \times s)_+, L(\mathcal{P}(X), t \times s)_+)$$

from Lemma 4.5 as described in Lemma 2.3 gives the dual pair in the lemma.  $\square$

Let  $f: X \rightarrow X$  be a  $G$ -map. Then  $f$  induces a map of spaces

$$\tilde{f}: \mathcal{P}(X) \rightarrow \mathcal{P}(X).$$

Note that

$$t \times s(\gamma, u) = (\gamma(u), \gamma(0)) \neq t \times s(f(\gamma, u)) = (f(\gamma(u)), f(\gamma(0))).$$

So  $\tilde{f}$  is not a map of ex-spaces. Let

$$\mathcal{P}^f X := \{(\gamma, u, x) \in \mathcal{P}X \times X \mid \gamma(0) = f(x)\}.$$

This is a space over  $X \times X$  via the map  $t \times s: \mathcal{P}^f X \rightarrow X \times X$  where  $s(\gamma, u, x) = x$  and  $t(\gamma, u, x) = \gamma(u)$ . Then

$$\tilde{f}: (\mathcal{P}X, s)_+ \rightarrow (\mathcal{P}X, s)_+ \odot (\mathcal{P}^f X, t \times s)_+.$$

is a map of right  $(\mathcal{P}X, t \times s)_+$ -modules.

Since  $(\mathcal{P}X, s)_+$  is dualizable, the trace of  $\tilde{f}$  is defined.

**Definition 12.2.** The *local Reidemeister trace* of  $f$ ,  $\mathcal{R}^l(f)$ , is the trace of  $\tilde{f}$ .

The invariant  $\mathcal{R}^l(f)$  is an element of  $\pi_0^{s,G} \left( \langle\langle (\mathcal{P}^f X, t \times s)_+ \rangle\rangle \right)$ . Like the invariants we have already defined, the local Reidemeister trace is an invariant of the equivariant homotopy class of  $f$ .

Let  $\Lambda_G^f X := \{\alpha \in X^I \mid \alpha(0) = f(\alpha(1))\}$ . Composition of paths defines an equivariant map from  $\langle\langle (\mathcal{P}^f X, t \times s)_+ \rangle\rangle$  to  $\Lambda_G^f X$  and this induces a map

$$\mu: \pi_0^{s,G} \left( \langle\langle (\mathcal{P}^f X, t \times s)_+ \rangle\rangle \right) \rightarrow \pi_0^{s,G} \left( \left( \Lambda_G^f X \right)_+ \right).$$

The decomposition in Theorem 13.3 can be used to show  $\mu$  is an isomorphism.

For an equivariant map  $f: X \rightarrow X$  and an open subset  $U$  of  $X$  there is an equivariant map

$$\Delta: X_+ \rightarrow (\bar{U}/\partial U) \wedge X_+$$

defined by

$$\Delta(x) = \begin{cases} (x, x) & \text{if } x \in \bar{U} \\ * & \text{if } x \notin \bar{U} \end{cases}$$

**Lemma 12.3.** *The local Reidemeister trace of an equivariant map*

$$f: X \rightarrow X$$

*is the equivariant stable homotopy class of the map*

$$S^V \xrightarrow{\text{tr}_\Delta(f_+)} S^V \wedge (U/\partial U) \xrightarrow{1 \wedge \iota} S^V \wedge \Lambda_G^f X_+.$$

The subspace  $U$  and map  $\iota$  were defined in Lemma 10.4. In this lemma we have used the isomorphism  $\mu$  above to identify  $\pi_0^{s,G} \left( \langle\langle (\mathcal{P}^f X, t \times s)_+ \rangle\rangle \right)$  and  $\pi_0^{s,G} \left( \left( \Lambda_G^f X \right)_+ \right)$ .

This map is the equivariant generalization of the Nielsen-Reidemeister index defined in [4, p. 207].

*Proof.* We can compare these maps explicitly.

Let

$$\chi: S^V \rightarrow T\nu$$

be the equivariant Pontryagin-Thom map of the normal bundle of  $M$  embedded in some representation  $V$ . Let

$$\epsilon: T\nu \wedge M_+ \rightarrow S^V$$

be the evaluation for the dual pair  $(M_+, T\nu)$ .

Then the local Reidemeister trace is the stable homotopy class of the map

$$S^V \xrightarrow{\chi} T\nu \longrightarrow (\mathcal{P}(X), s)_+ \odot T_X s^* S^V \xrightarrow{\tilde{f} \wedge \text{id}} (\mathcal{P}^f(X), s)_+ \odot T_X s^* S^V \longrightarrow S^V \wedge \Lambda_G^f X.$$

The image of a element  $v \in S^V$  under this map is

$$(\epsilon(\chi(v), fp\chi(v)), H(p\chi(v), fp\chi(v)))$$

where  $p$  is the quotient map  $\nu \rightarrow X$  and  $H$  is as in Lemma 10.4.

The image of  $v \in S^V$  under transfer of  $f$  is

$$(\epsilon(\chi(v), fp\chi(v)), p\chi(v)).$$

The image of  $v \in S^V$  under

$$S^V \xrightarrow{\text{tr}_\Delta(f_+)} S^V \wedge (U/\partial U) \xrightarrow{1 \wedge \iota} S^V \wedge \Lambda_G^f X.$$

is  $(\epsilon(\chi(v), fp\chi(v)), \iota(p\chi(v)))$ .  $\square$

### 13. COMPARISON OF REIDEMEISTER TRACES

In the previous three sections we have defined three versions of the Reidemeister trace. The first two versions, the geometric and global Reidemeister traces, detect fixed orbits in addition to fixed points. The third version, the local Reidemeister trace, only detects fixed points. In this section we will define geometric and global Reidemeister traces that only detect fixed points. We will also identify these invariants with the local Reidemeister trace.

The space  $\Lambda_G^f(X) = \{\gamma \in X^I \mid f(\gamma(0)) = \gamma(1)\}$  has an action of  $G$  defined using the action of  $G$  on  $X$ . The map

$$\oplus \left( \left( \Lambda_{WH}^{f^H} X^H \right) / WH \right) \rightarrow \langle\langle \Pi^f(G, X) \rangle\rangle$$

defined by  $\gamma \mapsto (R_e, [\gamma])$ , where  $[\gamma]$  denotes the equivalence class of  $\gamma$ , induces a map

$$\pi: \pi_0^s(\langle\langle \Pi^f(G, X) \rangle\rangle) \rightarrow \oplus \pi_0^s \left( \Lambda^{f^H} X^H / WH \right).$$

**Definition 13.1.** The *geometric Reidemeister trace* of  $f$ ,  $\mathcal{R}^g(f)$ , is the image of the (extended) geometric Reidemeister trace of  $f$  under  $\pi$ .

The following corollary is a consequence of Proposition 10.2.

**Corollary 13.2.** *For an equivariant map  $f: X \rightarrow X$  the geometric Reidemeister trace of  $f$  is*

$$\sum \frac{1}{|WH|} i(f|_{X^H}, \Theta^{-1}(R_e, \gamma)) [R_e, \gamma].$$

For a  $G$ -space  $Y$ , define a map

$$\xi_H: \pi_0(Y^H / WH) \rightarrow \pi_0^{s,G}(Y)$$

by  $\xi_H(f) = f_*(\text{tr}_\Delta(G/H_+))$ . We identify  $f \in \pi_0(Y^H / WH)$  with a  $G$ -map  $G/H \rightarrow Y$ .

**Theorem 13.3.** [6, III.8.13.7] *For a  $G$ -space  $Y$  the map*

$$\xi := \oplus \xi_H: \mathbb{Z} \left( \oplus \pi_0(Y^H / WH) \right) \rightarrow \pi_0^{s,G}(Y)$$

*is an isomorphism.*

**Proposition 13.4.** *The image of the geometric Reidemeister trace under the isomorphism*

$$\xi: \oplus \mathbb{Z} \pi_0(\Lambda^{f^H} X^H / WH) \rightarrow \pi_0^{s,G}(\Lambda_G^f X_+)$$

*in Theorem 13.3 is the local Reidemeister trace.*

*Proof.* Corollary 13.2 implies the geometric Reidemeister trace is

$$\sum \left( \frac{1}{|WH|} i(f|_{X^H \cap \alpha^{-1}(R_e, \gamma)}) \right) [(R_e, \gamma)].$$

Then  $\xi$  applied to the geometric Reidemeister trace is

$$\sum \left( \frac{1}{|WH|} i(f|_{X^H \cap \alpha^{-1}(R_e, \gamma)}) \right) (R_e, \gamma)_*(\text{tr}_\Delta(G/H_+)).$$



Since  $f$  is taut, we can use Lemma 10.4 to find an open set  $\tilde{V}_H$  in  $X^H$  that contains the fixed points of  $f_H$  and a map

$$\iota: \tilde{V}_H \rightarrow \Lambda_{WH}^f X^H$$

such that  $\iota(x)(0) = x$ ,  $\iota(x)(1) = f(x)$ , and  $\iota(x)(t) = x$  if  $x = f(x)$ . The map  $(R_e, \gamma): G/H \rightarrow \Lambda^f X$  factors as an inclusion

$$\overline{(R_e, \gamma)}: G/H \rightarrow \Pi V_g$$

followed by the map  $\iota$ .

The image of the geometric Reidemeister trace in  $\pi_0^{s,G}(\Lambda_G^f X_+)$  is then

$$\iota_* \left( \sum \left( \frac{1}{|WH|} i(f|_{X^H \cap \alpha^{-1}(R_e, \gamma)}) \right) \overline{(R_e, \gamma)}_* (\text{tr}_\Delta(G/H_+)) \right).$$

Then [19, III.8.4] identifies

$$\sum \left( \frac{1}{|WH|} i(f|_{X^H \cap \alpha^{-1}(R_e, \gamma)}) \right) \overline{(R_e, \gamma)}_* (\text{tr}_\Delta(G/H_+))$$

with the transfer of  $f$ . □

We can use a similar approach to define the global Reidemeister trace.

The inclusion

$$\Pi_{x(H) \in B(X)} \langle \pi_1(X^H, e) \rangle \rightarrow \langle \Pi_0^f(G, X) \rangle$$

induces a map

$$\pi: \mathbb{Z} \langle \Pi_0^f(G, X) \rangle \rightarrow \mathbb{Z} \left( \Pi_{x(H) \in B(X)} \langle \pi_1(X^H, e) \rangle \right).$$

This is compatible with the map  $\pi$  above.

**Definition 13.5.** The *global Reidemeister trace* of  $f$ ,  $\mathcal{R}^{gl}(f)$ , is the image of the (extended) global Reidemeister trace under  $\pi$ .

The following corollary is a consequence of Proposition 11.3.

**Corollary 13.6.** *There is an isomorphism*

$$\mathbb{Z} \pi_0(\Lambda^{f^H} X^H / WH) \cong \mathbb{Z} \Pi \langle \pi_1(X^H, x) \rangle.$$

*Under this isomorphism  $\mathcal{R}^g(f) = \mathcal{R}^{gl}(f)$ .*

The following corollary is a consequence of Proposition 11.5.

**Corollary 13.7.** *The global Reidemeister trace of  $f$  is*

$$\sum_{x(H) \in B(X)} q \left( \frac{1}{|WH|} \text{tr}_{\mathbb{Z}\pi_1(X^H)} \left( \left( C_* \left( \tilde{f} \right) : C_* \left( \widetilde{X^H}, \widetilde{X^{>H}} \right) \rightarrow C_* \left( \widetilde{X^H}, \widetilde{X^{>H}} \right) \right) \right) \right) [x(H)]$$

This is the definition of the generalized Lefschetz number in [31, 5.4].

## 14. THE CONVERSE TO THE EQUIVARIANT LEFSCHETZ FIXED POINT THEOREM

Fadell and Wong, [9], and Wilczyński, [33], have given proofs of the converse to the equivariant Lefschetz fixed point theorem. Here we describe the outline of an alternative proof due to Klein and Williams. See [15] for a complete proof. The results of this section and the previous section complete the proof of Theorem B.

We choose to use this proof to complete our approach to the equivariant converse since it easily generalizes and is very compatible with our definition of the local Reidemeister trace.

**Theorem 14.1.** *If  $X$  is a closed smooth  $G$ -manifold such that*

$$\dim(X^H) \geq 3 \text{ and } \dim(X^H) \leq \dim(X^K) - 2$$

*for all subgroups  $K \subsetneq H$  of  $G$  that are isotropy groups of  $X$ , the local Reidemeister trace of*

$$f: X \rightarrow X$$

*is zero if and only if  $f$  is equivariantly homotopic to a map with no fixed points.*

Let  $X$  be a closed smooth  $G$ -manifold and  $f: X \rightarrow X$  be an equivariant map. Let  $\Gamma_f: X \rightarrow X \times X$  be the graph of  $f$  and  $\Delta \subset X \times X$  be the diagonal. Factor the inclusion map  $i: X \times X \setminus \Delta \rightarrow X \times X$  as

$$X \times X \setminus \Delta \longrightarrow N_G(i) \xrightarrow{r_G(i)} X \times X$$

where the first map is an equivariant homotopy equivalence and  $r_G(i)$  is an equivariant Hurewicz fibration.

The unreduced fiberwise suspension of  $(\Gamma_f)_* r_G(i): (\Gamma_f)_* N_G(i) \rightarrow X$  is the double mapping cylinder

$$S_X((\Gamma_f)_* N_G(i)) := X \times \{0\} \cup_p ((\Gamma_f)_* N_G(i) \times [0, 1]) \cup_p X \times \{1\}.$$

This is a space over  $X$  and there are two sections  $\sigma_1, \sigma_2: X \rightarrow S_X(\Gamma_f)_* N(f)$  given by the inclusion of  $X \times \{0\}$  and  $X \times \{1\}$ .

If  $Z$  and  $W$  are ex- $G$ -spaces over  $X$  let  $\{Z, W\}_{G,X}$  be the fiberwise equivariant stable homotopy classes of maps from  $Z$  to  $W$ .

**Proposition 14.2.** [15, Corollary F] *Assume  $\dim(X^H) \geq 3$  and  $\dim(X^H) \leq \dim(X^K) - 2$  for all isotropy subgroups  $K \subset H$  of  $X$ . An equivariant map  $f: X \rightarrow X$  is equivariantly homotopic to a map with no fixed points if and only if*

$$\sigma_1 \amalg \sigma_2 \in \{S_X^0, S_X(\Gamma_f)_* N(i)\}_{G,X}$$

*is the trivial element.*

The proof of this proposition relies primarily on connectivity and dimension conditions that imply the unstable question of finding a fixed point free map can be identified with a stable question. To complete the proof of Theorem 14.1 we need to identify the class of  $\sigma_1 \amalg \sigma_2$  with the local Reidemeister trace.

**Proposition 14.3.** *There is an isomorphism*

$$\{S_X^0, S_X(\Gamma_f)_* N(i)\}_{G,X} \cong \pi_0^{s,G}(\Lambda^f X).$$

There are two steps in this proposition. The first is to give a different description of  $S_X(\Gamma_f)_* N(i)$ . The second step is an application of Costenoble-Waner duality, Definition 16.2.

Let  $\tau$  denote the equivariant tangent bundle of  $X$ .

**Lemma 14.4.** [15, 7.1] *There is an equivariant fiberwise weak equivalence*

$$S_{X \times X} N(X \times X - \Delta) \rightarrow \Delta_! S^\tau \boxtimes (\mathcal{P}X, t \times s)_+$$

over  $X \times X$ .

Recall that  $\boxtimes$  is the bicategory composition in the bicategory of parametrized  $G$ -spaces.

The tangent space of  $X$  can be identified with the normal bundle of the embedding of  $X$  in  $X \times X$  as the diagonal. Using this identification, a point in the tangent space corresponds to a point in a tubular neighborhood of the diagonal in  $X \times X$ . The image of

$$\sigma_1 \amalg \sigma_2 : S_X^0 \rightarrow S_X(\Gamma_f)_* N(f)$$

under the equivalence

$$S_X(\Gamma_f)_* N(f) \rightarrow (\Gamma_f)_*(\Delta_! S^\tau \boxtimes (\mathcal{P}X, t \times s)_+)$$

is

$$x \mapsto ((x, f(x)), \iota(x)).$$

The path  $\iota$  is defined in Lemma 10.4.

Using this weak equivalence we have an isomorphism

$$\{S_X^0, S_X(\Gamma_f)_* N(f)\}_{G, X} \cong \{S_X^0, (\Gamma_f)_*(\Delta_! S^\tau \boxtimes (\mathcal{P}X, t \times s)_+)\}_{G, X}.$$

Applying Costenoble-Waner duality, Proposition 16.3, and observing that the sum of the normal bundle and tangent bundle of  $X$  is the trivial bundle we get an isomorphism

$$\{S_X^0, (\Gamma_f)_*(\Delta_! S^\tau \boxtimes (\mathcal{P}X, t \times s)_+)\}_{G, X} \cong \pi_0^{s, G}(\Lambda^f X).$$

The image of  $\sigma_1 \amalg \sigma_2$  under this isomorphism is the map

$$S^V \xrightarrow{\chi} T\nu \xrightarrow{\Delta \wedge \iota} S^\nu \boxtimes (\Delta_! S^\tau) \boxtimes (\Gamma_f)_*(\mathcal{P}X, t \times s)_+ \xrightarrow{\cong} S^V \wedge (\Gamma_f)_*(\mathcal{P}X, t \times s)_+.$$

The map  $\chi$  is the Pontryagin-Thom map for the embedding of  $X$  in  $V$ . If  $p: \nu \rightarrow M$  is the projection map,  $\Delta \wedge \iota$  takes  $v \in T\nu$  to

$$(v, (p(v), f(p(v))), \iota(p(v))).$$

If  $\epsilon$  is the evaluation for the dual pair in Proposition 1.5, the image of

$$(v, p(v), f(p(v)), \iota(p(v)))$$

in

$$S^V \wedge (\Gamma_f)_*(\mathcal{P}X, t \times s)_+$$

is  $(\epsilon(f(p(v)), v), \iota(p(v)))$ .

**Proposition 14.5.** *If  $X$  is a compact smooth  $G$ -manifold and  $f: X \rightarrow X$  is an equivariant map, the image of*

$$\sigma_1 \amalg \sigma_2 \in \{S_X^0, S_X(\Gamma_f)_* N(f)\}_{G, X}$$

in

$$\pi_0^{s, G}(\Lambda^f X_+)$$

is the local Reidemeister trace.

*Proof.* Let  $\phi_G$  be the weak equivalence of Lemma 14.4. Let  $F$  be stabilization and  $D$  be the isomorphism that defines Costenoble-Waner duality, Proposition 16.3. Then the following diagram of isomorphisms commutes.

$$\begin{array}{ccc}
[S_X^0, S_X f^* N_G(X \times X - \Delta)]_X & \xrightarrow{\phi_{G*}} & [S_X^0, \Delta_! S^{\tau_G} \boxtimes \Lambda^f X_+]_X \\
\downarrow F & & \downarrow F \\
\{S_X^0, S_X f^* N_G(X \times X - \Delta)\}_X & \xrightarrow{\phi_{G*}} & \{S_X^0, \Delta_! S^{\tau_G} \boxtimes \Lambda^f X_+\}_X \\
\downarrow D & & \downarrow D \\
\{S^V, S_X^\nu \boxtimes S_X f^* N_G(X \times X - \Delta)\}_{(\text{id} \boxtimes \phi_G)_*} & \xrightarrow{\quad} & \{S^V, S_X^\nu \boxtimes \Delta_! S^{\tau_G} \boxtimes \Lambda^f X_+\}
\end{array}$$

The image of  $\sigma_1 \amalg \sigma_2$  in the bottom right corner is the map

$$S^V \rightarrow S^V \wedge \Lambda^f X_+$$

defined by

$$v \mapsto (\epsilon(f(p(\chi(v))), \chi(v)), \iota(p(\chi(v)))).$$

Lemma 12.3 implies this is the local Reidemeister trace.  $\square$

**Theorem B.** *Let  $X$  be a closed smooth  $G$ -manifold such that*

$$\dim(X^H) \geq 3 \text{ and } \dim(X^H) \leq \dim(X^K) - 2$$

*for all subgroups  $K \subsetneq H$  of  $G$  that are isotropy groups of  $X$ . Then*

$$f: X \rightarrow X$$

*is equivariantly homotopic to a map with no fixed points if and only if  $\mathcal{R}^{gl}(f) = 0$ .*

*Proof.* Proposition 14.2 shows  $f$  is equivariantly homotopic to a map with no fixed points if and only if  $\sigma_1 \amalg \sigma_2$  is trivial. Proposition 14.5 shows  $\sigma_1 \amalg \sigma_2$  is trivial if and only if  $\mathcal{R}^l(f)$  is trivial. Proposition 13.4 shows  $\mathcal{R}^l(f) = \mathcal{R}^g(f)$ . Proposition 11.3 shows  $\mathcal{R}^g(f) = \mathcal{R}^{gl}(f)$ .  $\square$

## 15. EQUIVARIANT NIELSEN NUMBERS

If there is no group action, the Nielsen number is the number of nonzero coefficients in the Reidemeister trace. In the equivariant generalizations of these invariants this connection does not hold, but it remains true that the Nielsen number is zero if and only if the Reidemeister trace is zero. We start by considering an analogous result for the Lefschetz number.

If  $X$  is compact, the spaces  $X^H$  are compact for each subgroup  $H$  of  $G$  and the (nonequivariant) index of  $f^H: X^H \rightarrow X^H$  is defined.

**Proposition 15.1.** *All integers in the set  $(i(f^H|_{X^H(x)}))_{H \subset G}$  are zero if and only if the local Lefschetz number is zero.*

Recall  $i$  denotes the nonequivariant index.

*Proof.* This follows from Proposition 6.2 and Proposition 1.6.

If the local Lefschetz number of  $f$  is zero then Proposition 6.2 implies

$$i(f_H|_{X_H(x)}) = 0$$

for all  $x(H) \in B(x)$ . Let  $\{H_1, H_2, \dots, H_n\}$  be the conjugacy classes of subgroups of  $G$ . We require that  $H_i \subset H_j$  implies  $i \leq j$ . In particular,  $H_n = G$  and  $H_1$  consists only of the identity element.

Since  $X^G = X_G$ ,  $i(f^G|_{X^G(x)})$  is zero. The map is taut, so Proposition 1.6 implies

$$i\left(f^{H_{n-1}}|_{X^{H_{n-1}}(x)}\right) = i\left(f^{H_{n-1}}|_{X^{H_{n-1}}(x)}\right) + i\left(f^{H_{n-1}}|_{X^{>H_{n-1}}(x)}\right).$$

The partial order implies  $X^{>H_{n-1}} = X^G$ . Since  $i(f^G|_{X^G(x)})$  and  $i(f^{H_{n-1}}|_{X^{H_{n-1}}(x)})$  are both zero,  $i(f^{H_{n-1}}|_{X^{H_{n-1}}(x)})$  is also zero. The remaining steps in the induction are similar.

For the converse, assume  $i(f^H|_{X^H(x)})$  for each  $x(H) \in B(X)$ . A similar argument shows  $i(f^H|_{X^H(x)})$  is zero for each  $x(H) \in B(X)$ .  $\square$

To define the equivariant Nielsen number we need to introduce fixed point classes. First consider the relation defined by the map  $\Theta$  in Section 10.

**Lemma 15.2.** *Two points  $y$  and  $z$  of  $\mathcal{O}(f)$ , such that  $f(y) = ym$  and  $f(z) = zn$ , have the same image under  $\Theta$  if and only if  $G_z$  is conjugate to  $G_y$ , there is an element  $p$  of  $WH$  such that  $mp = pn$ , and there is a path  $\psi$  from  $z$  to  $yp$  such that  $\psi n$  is homotopic to  $f(\psi)$ .*

*Proof.* If the points  $y$  and  $z$  have the same image under  $\Theta$  there are elements  $g, h \in WH$ , a path  $\delta$  from  $f(y)$  to  $zg$ , and a path  $\gamma$  from  $z$  to  $yh$ , such that  $(\gamma g)\delta$  is homotopic to the constant path at  $f(y) = ym$  and  $(\delta h)f(\gamma)$  is homotopic to the constant path at  $f(z) = zn$ . Then the path  $\gamma$  is a path from  $z$  to  $yh$  and

$$f(\gamma) \simeq (\gamma gh)(\delta h)f(\gamma) \simeq \gamma gh.$$

For the converse suppose there is a path  $\psi$  from  $z$  to  $zp$  such that  $\psi n$  is homotopic to  $f(\psi)$  with end points fixed. The image of  $y$  under  $\Theta$  is  $(R_m, c_{f(y)})$ . Since  $\psi n$  is homotopic to  $f(\psi)$  the constant path at  $f(y)$  can be replaced by

$$(\psi p^{-1}n) f(\psi^{-1}p^{-1}).$$

This is identified with the path

$$(\psi^{-1}n)(\psi n)$$

which is homotopic to the constant path at  $f(z) = zn$ . This path is the image of  $z$  under  $\Theta$ .  $\square$

For each subgroup  $H$  of  $G$  define a map  $\theta$  from the fixed points of  $f^H$  to  $\langle \pi_1(X^H) \rangle$  by

$$\theta(x) = \gamma^{-1}f(\gamma)\tau$$

where  $\gamma$  is any path from  $*$  to  $x$  and  $\tau$  is a fixed path from  $*$  to  $f(*)$ . Two fixed points  $x$  and  $y$  in  $X^H$  are in the same  $(H)$ -fixed point class if  $\theta(x) = \theta(y)$ . We will modify this relation a little to define equivariant Nielsen numbers. The most significant change is that we will relax the hypothesis that  $G_z$  is conjugate to  $G_y$ .

If  $K$  is subconjugate to  $H$  there is a map  $X^H \rightarrow X^K$ . There is also a map

$$\Phi: \langle \pi_1(X^H) \rangle \rightarrow \langle \pi_1(X^K) \rangle.$$

If  $\alpha \in \langle \pi_1(X^K) \rangle$  let

$$\Phi^{-1}(\alpha) = \left\{ \gamma \in \prod_{H < K} \langle \pi_1(X^H) \rangle \mid \Phi(\gamma) = \alpha \right\}.$$

**Definition 15.3.** [34] The *equivariant Nielsen number* of  $f$ ,  $N_G(f)$ , is a function from the conjugacy classes of subgroups of  $G$  to the integers defined by

$$N_G(f)(H) = \# \left\{ \alpha \in \langle \pi_1(X^H) \rangle \mid \begin{array}{l} i(f^H, \theta^{-1}(\alpha)) \neq 0 \text{ and} \\ i(f^K, \theta^{-1}(\delta)) = 0 \text{ for all } \delta \in \Phi^{-1}(\alpha) \end{array} \right\}$$

Here  $i$  denotes the nonequivariant fixed point index.

**Proposition 15.4.** *The equivariant Nielsen number of a map is zero if and only if the equivariant geometric Reidemeister trace is zero.*

*Proof.* First note that  $\theta^{-1}(\alpha)$  is

$$\prod_{\delta \in \Phi^{-1}(\alpha)} \Theta^{-1}(R_e, \delta).$$

Since the map is taut and the index is additive, [30, III.5.3],

$$i(f|_{X^H}, \theta^{-1}(\alpha)) = \sum_{\delta \in \Phi^{-1}(\alpha)} i(f|_{X^H}, \Theta^{-1}(R_e, \delta)).$$

Using Corollary 13.2, the geometric Reidemeister trace of  $f$  is

$$\sum \left( \frac{i(f|_{X^H}, \Theta^{-1}(R_e, \gamma))}{|WH|} \right) [(R_e, \gamma)].$$

The geometric Reidemeister trace of  $f$  is zero if and only if  $i(f|_{X^H}, \Theta^{-1}(R_e, \gamma)) = 0$  for each  $(R_e, \gamma)$ .

If the geometric Reidemeister trace is zero then  $i(f|_{X^H}, \theta^{-1}(\alpha)) = 0$  for all  $\alpha$ . Since all of the indices are zero, the Nielsen number is zero.

If the Nielsen number is zero, a similar argument using the order on the subgroups from Proposition 15.1 shows the Reidemeister trace is also zero.  $\square$

*Remark 15.5.* Note that the equivariant Nielsen number is *not* the number of generators in  $\mathcal{R}^g(f)$  with nonzero coefficient. The equivariant Nielsen number is a ‘non-redundant’ count of the number of nonzero coefficients. In particular, the coefficients of  $\mathcal{R}^g(f)$  do not give a lower bound for the number of fixed points.

## 16. A REVIEW OF PARAMETRIZED SPACES

Through out this paper and especially in Sections 3, 6, 10, and 12 we freely used the results, definitions, and notation of [23]. In this section we recall the notation we use and some of the results. For more complete descriptions and examples from cases with no group action see [24].

An *ex- $G$ -space*  $X$  over a  $G$ -space  $B$  is a  $G$ -space  $X$  with equivariant maps  $s: B \rightarrow X$  and  $p: X \rightarrow B$  such that  $p \circ s$  is the identity map of  $B$ .

If  $f: B \rightarrow C$  is a  $G$ -map,  $f_!X$  is the ex- $G$ -space over  $C$  with total space the push out

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ \downarrow & & \downarrow \\ X & \longrightarrow & f_!X \end{array} .$$

The universal property of the pushout defines a map  $f_!X \rightarrow C$ . If  $Y$  is an ex- $G$ -space over  $C$ ,  $f^*Y$  is the ex- $G$ -space over  $B$  with total space the pull back

$$\begin{array}{ccc} f^*Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array} .$$

The universal property of the pullback defines a map  $B \rightarrow f^*Y$ .

If  $X$  is an ex- $G$ -space over  $B$  and  $Y$  is an ex- $G$ -space over  $C$ ,  $X \bar{\wedge} Y$  is an ex- $G$ -space over  $B \times C$  and the fiber over a point  $(b, c) \in B \times C$  is the smash product of the fiber of  $X$  over  $b$  with the fiber of  $Y$  over  $c$ .

If  $X \in G\text{Ex}(A, B)$  and  $Y \in G\text{Ex}(B, C)$  are 1-cells,  $X \boxtimes Y$  is the pullback and then pushforward in the following diagram.

$$\begin{array}{ccccc} A \times C & \longleftarrow & A \times B \times C & \longrightarrow & A \times B \times B \times C \\ \downarrow & & \downarrow & & \downarrow \\ X \boxtimes Y & \longleftarrow & \Delta_*(X \bar{\wedge} Y) & \longrightarrow & X \bar{\wedge} Y \\ \downarrow & & \downarrow & & \downarrow \\ A \times C & \longleftarrow & A \times B \times C & \longrightarrow & A \times B \times B \times C \end{array}$$

The product  $\boxtimes$  is the bicategory composition in the bicategory  $G\text{Ex}$ . This bicategory has 0-cells  $G$ -spaces and 1-cells ex- $G$ -spaces. The 2-cells are fiberwise maps that are also equivariant.

*Remark 16.1.* We will require that the base and total spaces of the 1-cells in  $G\text{Ex}$  are of the homotopy type of  $G$ -CW-complexes, the projection is an equivariant Hurewicz fibration, and the section is an equivariant fiberwise cofibration. When these conditions are satisfied the constructions we use produce the correct homotopy type. The category with objects ex-spaces that satisfy these conditions and morphisms equivariant fiberwise homotopy classes of maps is equivalent to the homotopy category that is defined using model categories in [23]. If these conditions are not satisfied we will use the approximations described in [23].

In Section 2 we defined duality for bicategories. There is one example of duality in the bicategory  $G\text{Ex}$  that we used frequently.

**Definition 16.2.** [23, 18.3.1] Let  $X$  and  $Y$  be 1-cells in  $G\text{Ex}$  where  $X$  is an ex-space over  $* \times B$  and  $Y$  is an ex-space over  $B \times *$ . We say  $(X, Y)$  is a *Costenoble-Waner  $V$ -dual pair* if there are maps

$$\eta: S^V \rightarrow X \boxtimes Y$$

and

$$\epsilon: Y \boxtimes X \rightarrow \Delta_! S_B^V$$

such that

$$\begin{array}{ccc}
 S^V \wedge X & \xrightarrow{\eta \wedge \text{id}} & X \boxtimes Y \boxtimes X \\
 & \searrow \gamma & \downarrow \text{id} \boxtimes \epsilon \\
 & & X \wedge S^V
 \end{array}
 \qquad
 \begin{array}{ccc}
 Y \wedge S^V & \xrightarrow{\text{id} \wedge \eta} & Y \boxtimes X \boxtimes Y \\
 & \searrow \gamma(\text{id} \boxtimes \alpha) & \downarrow \epsilon \boxtimes \text{id} \\
 & & S^V \wedge Y
 \end{array}$$

commute up to equivariant stable homotopy.

Recall that  $\gamma$  is the symmetry isomorphism. The map  $\alpha$  takes  $v$  to  $-v$ .

There is another characterization of dualizable objects. If  $X$  and  $Y$  are parametrized  $G$ -spaces over a  $G$ -space  $M$ , let  $\{X, Y\}_{G, M}$  denote the equivariant fiberwise sectioned stable homotopy classes of maps from  $X$  to  $Y$ .

**Proposition 16.3.** [23, 16.4.6] *An ex-space  $X$  over  $* \times B$  is Costenoble-Waner dualizable with dual  $Y$  if and only if the map*

$$\eta/(-): \{W \boxtimes X, U\}_{G, M \times B} \rightarrow \{W, U \boxtimes Y\}_{G, M}$$

defined by

$$\eta(f): W \boxtimes S^V \xrightarrow{\text{id} \odot \eta} W \boxtimes X \boxtimes Y \xrightarrow{f \odot \text{id}} U \boxtimes Y$$

is an isomorphism for all ex-spaces  $U$  over  $M \times B$  and  $W$  over  $M$ .

The following proposition corresponds to Proposition 1.5 in Section 1.

**Proposition 16.4.** [23, 18.5.1, 18.6.1]

- (1) *If  $K$  is a compact  $G$ -ENR embedded in a representation  $V$ , then  $(S_K, C_K(V \setminus K))$  is a Costenoble-Waner  $V$ -dual pair.*
- (2) *If  $M$  is a smooth closed manifold embedded in a representation  $V$  with equivariant normal bundle  $\nu_M$  then  $(S_M, S^{\nu_M})$  is a Costenoble-Waner  $V$ -dual pair.*

The ex-space  $C_K(V \setminus K)$  is the cone of the inclusion  $V \setminus K$  into  $K$ . The ex-space  $S^{\nu_M}$  is the fiberwise one point compactification of  $\nu_M$ .

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